GRE Physics: Comprehensive Notes

Gabriele Montefalcone

Contents

1 Classical Mechanics

1.1 Dynamics

- $\mathbf{F}_{\text{tot}} = m\mathbf{a}$; $F_{1-2} = F_{2-1} \rightarrow$ for an object not to accelerate in direction x, all forces on x sum up to zero
- think always of *limiting cases* (*e.g.* if I know $F = 0$ at $\theta = 0$ and $F = 1$ at $\theta = \frac{\pi}{2}$ probably $\propto \sin(\theta)$)
- $F_s \leq \mu F_N$ (static friction) $\rightarrow$ if body *just* moves the friction force is max and equals kinetic friction ($F_s^{\max} = F_d$)
- if two objects are:
  - *not* distinct, they move with the same $\mathbf{a}$ (*e.g.* one on top of the other, even if attached by a massless spring in between there would be no tension)
  - *independent*, then consider the forces separately (if friction is involved this is always the case)
- projectile motion: $x(t) = V_{0,x}t + x_0$ (no force acting so $V_{0,x} = \text{cost}$)
  - $y(t) = y_0 + V_{0,y}t - \frac{g}{2}t^2$ (since $a = -g$)
  - Recall! from kinematics $v_f^2 - v_i^2 = 2a\Delta s$ where $\Delta s = s_f - s_i$
- uniform circular motion $\rightarrow a = \frac{v^2}{R}$; $v = \omega r = \frac{2\pi R}{T}$; $T = \frac{2\pi}{\omega}$; $f = \frac{\omega}{2\pi} = \frac{1}{T}$ (don't forget this! check units)
  - F $= \frac{mv^2}{R} \rightarrow$ valid also if *not* uniform but only when all forces are *Radial*!
    (*e.g.* pendulum at lowest point $\mathbf{F}_{\text{tot}} = T - mg = \frac{mv^2}{R}$)
  - if body does not have constant tangential $v \rightarrow \mathbf{a}$ must have also a tangential component
- To compute terminal velocity $F_g = F_d$ (where F_d is the drag force $\propto v$)
- $\Delta E = \Delta K + \Delta U = \Delta W_{\text{NC}}$ where W_{NC} is work done by non conservative forces
  - $\Delta E = 0$ if all forces are conservative
  - $\Delta W_{\text{tot}} = \Delta W_C + \Delta W_{\text{NC}} = \Delta K$
  - $U_e = \frac{1}{2}k\Delta x^2$ (elastic potential energy)
  - $K = K_T + K_R$ where $K_T = \frac{1}{2}mv^2$ (transational); $K_R = \frac{1}{2}I\omega^2$ (rotational)
- F $= \dot{p}$ so if $\mathbf{F}^{\text{ext}} = 0$; in collisions momentum is *conserved* ($p = mv = \text{const}$)
  - *elastic*: also conservation of total energy (*never* assume this unless stated!)
  - completely *inelastic*: both particles stick together post collision
  - $\Delta p = F \cdot \Delta t = \mathbf{J}$ (impulse)
  - time avg. force: $\bar{F}_t = \frac{1}{T}\int_{t_i}^{t_f} F dt = \frac{\text{J}}{T} = \frac{\Delta p}{T}$
  - distance avg. force: $\bar{F}_d = \frac{1}{D}\int_{d_i}^{d_f} F dx = \frac{\Delta W}{D} = \frac{\Delta K}{D}$
- if you have two plots of v_x and v_y vs $t \rightarrow$ to determine angle check ratio between v_x^0 and v_y^0.
- minimum to complete 1 revolution is $v = 0$ at peak (use energy conservation and recall that at peak $E = U$)
- to calculate deflection angle give eq. of motion in x, y we know $\tan(\theta) \approx \frac{dy}{dx}$.
- When changing reference frame think *very carefully* of where you will be and let intuition guide you
- to know how *fast* and how *far*: energy conservation; to know how much *time*: kinematics
- Rocket motion: $m\frac{dv}{dt} + u\frac{dm}{dt} = F_{\text{tot}}^{\text{ext}}$ (if no ext. forces left side is *conserved*)
  - rocket exhaust velocity u is taken relative to the rocket

1.2 Rotations

- Rolling without slipping: $v = \omega R$; $a = \alpha R$; $K_{\text{tot}} = K_{\text{tr}} + K_{\text{rot}} = \gamma m v^2$
  - point of contact with surface has always *zero* relative velocity
  - friction is the cause but it does *no* work (with no friction bodies would just slide)
- moment of inertia $I = \int r^2 dm = \int r^2 \rho dV$ → Recall! r is distance from axis of rotation (not just origin)
  - $\frac{1}{12}Ml^2$ (rod); $\frac{1}{2}MR^2$ (disk-cylinder); $\frac{2}{5}MR^2$ (sphere)
  - $I = I_{\text{CM}} + MR^2$ (parallel axis theorem)
  - if you have multiple objects attached to each other: *sum* individual Is as computed from the pivot
  - center of mass $r_{\text{CM}} = \frac{1}{M}\int r dm = \frac{1}{M}\int r \rho dV$
- angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p} = I\omega$ (if object is both rotating and translating then $\mathbf{L}_{\text{tot}} = \mathbf{L}_{\text{tr}} + \mathbf{L}_{\text{rot}}$
  - $\tau = \frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}$ (torque) → if $\tau_{\text{tot}}^{\text{ext}} = 0$ angular momentum is conserved
  - if tension acts *radially* $\mathbf{L} = \text{const}$, so we can have instances where E changes (tension does work to decrease radius) and $\mathbf{L}$ is conserved
  - $\mathbf{F}_{\text{tot}} = 0$ does *not* imply that $\mathbf{L}$ is conserved; it only means $\mathbf{a}_{\text{tot}} = 0$!
- if reference frame is rotating with constant angular velocity Ω it's *not* inertial!
  - We must add to $\mathbf{F}_{\text{tot}} = m\mathbf{a}$ two terms: $\mathbf{F}_{\text{centrifugal}} = -m\Omega^2\mathbf{r}$ (apparent force against centripetal)
  - $\mathbf{F}_{\text{coriolis}} = -m\mathbf{\Omega} \times \mathbf{v}$ (only exists if object is non-stationary in rotating frame)
- for merry-go arounds and spinning disks problems angular momentum conserved → $L_i = I_i\omega_i = I_f\omega_f = L_f$

1.3 Lagrangians

- $\mathscr{L} = T - U$ → most important step is to find coordinates that define movement of body the best
- $E - L$ eqs: $\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) = \frac{\partial \mathscr{L}}{\partial q}$ where $p = \frac{\partial \mathscr{L}}{\partial \dot{q}}$ (momentum conjugate)
- if $\frac{\partial \mathscr{L}}{\partial q} = 0$ → p is conserved.
- $\mathscr{H} = \sum_i p_i q_i - \mathscr{L} = T + U$ (if U not explicitly dependent on $\dot{q}_i$ and t); $\dot{p} = -\frac{\partial \mathscr{H}}{\partial q}$; $\dot{q} = \frac{\partial \mathscr{H}}{\partial p}$

1.4 Orbits

- With *central* forces $\mathbf{L} = \text{const}$ so motion confined to a plane with $l = mr^2\dot{\phi}$ → $\boxed{\mathscr{L} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - U(r)}$
- $F_{\text{gr}} = -\frac{GMm}{r^2}$; $U(r) = -\frac{GMm}{r}$; $V_{\text{eff}} = \frac{l^2}{2mr^2} + U(r)$
- With two bodies we use the same $\mathscr{L}$ with the $m \to \mu$ where μ is the reduced mass equal to $\frac{m_1 m_2}{m_1 + m_2}$
- with multiple gravitational masses $F_{\text{tot}} = \sum_i F_i^{\text{gr}}$ and from there get effective mass!
- $E_{\text{tot}} > 0$ (hyperbola-open); $E_{\text{tot}} = 0$ (parabola-open); $E_{\text{tot}} < 0$ (ellipse-bounded); $E_{\text{tot}} = V_{\text{min}}$ (circle-bounded)
- To find orbit radius set $\dot{V_{\text{eff}}(r)} = 0$ (stable equilibrium if $\ddot{V_{\text{eff}}(r)} \geq 0$)
  - stable non-circular orbits can *only* occur for simple harmonic potential and the inverse-square law force
  - given $F \propto r^{-n}$: for $n < 3$ a stable circular orbit *always* exists
  - bound orbits do not mean *closed*: they simply oscillate between two radii
  - distance of closest approach is when $\dot{r} = 0$ ($E = V(r)$) → watch out for what a distance is: if Sun is at 1 focus, $r \neq a$ which is the semi-major axis.
  - to determine shape of orbit compare its velocity to $v_{\text{esc}} = \sqrt{\frac{2GM}{r}}$ and $v_{\text{circ}} = \sqrt{\frac{GM}{r}}$

- for $r \to \infty$ $E_{r\to\infty} = K$; at closest approach $E_{r_{\min}} = V$ and since energy is conserved: $E_{r\to\infty} = E_{r_{\min}}$

- Kepler's Laws are: (I) planets are on elliptical orbits with sun at 1 focus (assumption $M_\odot \ggg m_p$);
  (II) orbits span equal areas in equal times $\to \frac{l}{m}dt = r^2 d\phi = dA$: $\frac{dA}{dt} = \frac{l}{m}$;
  (III) $T = ka^{\frac{3}{2}}$ where a is semi-major axis of orbit and $k = \frac{2\pi}{\sqrt{G(m_p+M_\odot}} \approx \frac{2\pi}{\sqrt{GM_\odot}}$

1.5 Springs

- $F_e = -kx \to$ if springs connected $k_{\text{tot}} = \sum_i k_i$ (in parallel); $1/k_{\text{tot}} = \sum_i 1/k_i$ (in series)
- for spring problems always (I) try limit cases first (dimensional analysis/symmetry); (II) try conservation of energy and (III) as a last resort try to solve differential eq.
- S.H.0.: $\omega = \frac{k}{m}$; damped oscillators have additional damping term $F_{\text{damp}} = -b\dot{x}$ s.t.: $\boxed{m\ddot{x} + b\dot{x} + kx = 0}$.
  - underdamped: exponentially *decaying* oscillations $\to \omega_1^2 = \omega_0^2 - \beta^2$ with $\omega_0^2 = k/m$; $\beta = b/2m$;
  - overdamped: *no* oscillation, just exponential decay
- driven Oscillator: guess complex solution $Ae^{i\omega t}$ where ω is driven frequency $\to A \propto 1/\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$
  - $A^{\max}$ at $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$ (resonance); with *no damping* $A \propto 1/|\omega_0^2 - \omega^2|$
- For more than a spring, consider matrix of eqs. of motions from lagrangian analysis: $\sum_k (A_{jk}q_k + m_{jk}\ddot{q_k}) = 0$
  - guess $q_k = a_k e^{i\omega t} \to$ solve $\det(A_{jk} - m_{jk}\omega^2) = 0$ (diagonalize the matrix) which gives the n frequencies ω_i at which system oscillates
  - # of normal ω_i = # of independent variables needed to describe system
  - always ask yourself what are the simplest ways for a system to oscillate $\to$ symmetric motion always lower frequency than antisymmetric motion
  - *lowest* collective motion has $\omega^2 = k_{\text{eff}}/m_{\text{eff}}$; *highest* motion is all out of phase with $\omega = \sum_i \omega_i$
- anything can be analyzed as an oscillation if we perturb system *only slightly* from its eq. of motion
- Recall for $\theta \lll 1 \to [\sin\theta, \cos\theta, \tan\theta] \approx [\theta, 1 - \theta^2/2, \theta]$.

1.6 Fluid Mechanics

- $P = \frac{dF}{dA}$ (pressure) $\to F = \int P dA$ where dA is the *cross-sectional* area
- given fluid at rest, pressure as function of height is $p - p_0 - = \rho g h$ (height of fluid on top of reference point P)
- equipressure means at given height pressure is the same!
- Bernoulli's principle $\frac{v^2}{2} + gz + \frac{p}{\rho} =$ const! (kind of conservation of energy eq.)
- fluid going through sectional area A_i with velocity v_i is *conserved*: $\rho \Delta t A_i v_i =$ const $\Rightarrow A_i v_i$ is conserved if ρ is uniform and equal time
- Boyant force: $F_b = \rho V g$ ($\uparrow$ upward direction) where ρ is the density of space where object is confined and V is the volume of the object:
  - if an object *galleggia* $F_g = F_b$
  - to lift a body from water the force required is $F_{\text{lift}} = F_g - F_b$

2 Electricity & Magnetism

2.1 Electrostatics

- $\nabla \cdot \vec{E} = \rho/\epsilon_0$; $\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla V$ with $V = -\int_a^b \vec{E} \cdot d\vec{l}$ (defined relative to some location)
  - V is usually set to zero at infinity; both $\vec{E}$ and V are additive
  - $\nabla^2 V = -\rho/\epsilon_0$: $V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r)}{|r-r'|} d^3r' \rightarrow$ from here compute $\vec{E}$
  - $\vec{F} = q\vec{E}$: per unit volume ($\rho\vec{E}$); area ($\sigma\vec{E}$)
  - to compute ρ use $\nabla \cdot \vec{E} = \rho/\epsilon_0$ (specifically if $\vec{E}$ is given in *vector* form)
- Gauss's law: $\boxed{\oint_S \vec{E}(r) \cdot d\vec{s} = \frac{Q_{\text{enc}}}{\epsilon_0}}$ $\Rightarrow$ use for *spherical* ($\sim \frac{1}{r^2}$); *cylindrical* ($\sim \frac{1}{r}$); *planar* ($\sim$ const) symmetries
- Recall: always start by *symmetry* considerations to limit computation:
  - *point* charges: just sum up potential from single configurations and take *derivative* to find $\vec{E}$ $\rightarrow$ if point charges *vertices* of polygon, then field and force at the center is *zero*
  - infinite *plane* with surface charge σ: $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$; infinite *line/cylinder* with charge per length λ: $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$
- V is always *continuous*, derivatives of V too except at surface charges $\rightarrow$ BC$_s$: $\boxed{E^{\parallel}_{\text{out}} - E^{\parallel}_{\text{in}} = 0; E^{\perp}_{\text{out}} - E^{\perp}_{\text{in}} = \frac{\sigma}{\epsilon_0}}$
- image method (to determine V): place image charge at point given by *reflection* about plane
  - with plane: on opposite side at *same* distance
  - charges add up to the right amount in each region where we compute potential $\rightarrow$ Recall to compute $\vec{E}$ field directly from configuration (*do not* take derivative of potential)
  - Recall: force on a charge is the same as that given by image charges $\rightarrow$ work is only done on *real* charges (no energy cost to move image charges)
- Conductors: *inside* $\vec{E} = 0$;$\rho = 0$; $V =$ const $\rightarrow$ induce opposite charge and leave all charges at surface $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$
  - resistivity: $\rho = \rho_0(1 + \alpha\Delta T) \propto \Delta T$; conductivity: $\sigma \propto 1/\Delta T$ (for $T \uparrow \rightarrow \infty$)
- Semiconductors: *decreasing* electrical resistivity with *increasing* temperature: $\rho \downarrow$ continuously
  - current conduction via *mobile* electrons which are forbidden from being excited until they overcome *band-gap*
  - for $T \uparrow$ electrons overcome band-gap and are *free* from constraints of exclusion principle
  - doped materials: excess of *holes*: $p-$type; of *electrons*: $n-$type
  - Diode is $p - n$ junction where I flows only if $V_{\text{app}} > V_{\text{bias}}$ and is *independent* of V_{app}!
    $\rightarrow$ Recall: a diode *blocks* the current in one direction and allows it in the other!
- $W = \frac{1}{2}\sum_i q_i V(r_i) = \frac{1}{2}\int \rho(r)V(r)d^3r$: work required to put together n charges (*negative*!)
  $\rightarrow$ sometimes convenient to think of work to move charges as $W = q\Delta V = q(V_f - V_i)$
- Energy *stored* in electric field is $U_E = \frac{\epsilon}{2}\int |E|^2 d^3r \rightarrow$ to find the work done to obtain a configuration $W = U_E^1 - U_E^2$ (always take difference and then perform integral!)
- Recall: *superposition* principle applies to $\vec{F}, \vec{E}, V, W$
- Capacitors: $Q = CV$ where capacitance C depends on *geometry* of problem (for parallel plate capacitor $C = \frac{\epsilon_0 A}{d}$)
  - $U_C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2$ (energy stored);
  - if two conductors touch, they become *equipotential* and if $C_1 = C_2 = C$ then charge equally distributes
  - if capacitors are connected s.t. opposite plates face each other: $Q_{\text{tot}} = Q_1 - Q_2 \neq Q_1 + Q_2$
- total flux given electric field is always by Gauss's Law $Q_{\text{tot}}/\epsilon_0$ then think how many field lines hit the plane

2.2 Magnetostatics

- $\nabla \cdot \vec{B} = 0$ (no magnetic monopoles); $\nabla \times \vec{B} = \mu_0 \vec{J} \to \vec{B} = \nabla \times \vec{A}$
  - $\vec{B}$ always *circles* around currents; $\vec{B} \to 0$ far away from current sources
  - $\vec{F} = q\vec{v} \times \vec{B}$; on current I: $d\vec{F} = Id\vec{l} \times \vec{B}$ s.t. $\vec{F}_{1,2} = LI_2 \times \vec{B}_1$ (force on 2 by 1)
  - for *cross* products: cylindrical $\boxed{\hat{z} \times \hat{r} = \hat{\phi}}$; spherical: $\boxed{\hat{\phi} \times \hat{r} = \hat{\theta}}$ $\Rightarrow$ other relations by *cyclic* permutations
  - $d\vec{K} = Id\vec{w}$ (surface current); $d\vec{J} = IdA$ (volume current)
  - $\vec{J} = \sigma\vec{E} = nqv$ and σ conductivity; v drift velocity; n charge density (concentration)
  - $\vec{B}$ do *no* work since $\vec{F} \perp \vec{v}$ (can only change direction of motion, not magnitude of velocity)
    $\to$ energy still stored in $\vec{B}$ field: $U_B = \frac{1}{2\mu_0}\int |B|^2 d^3r$
- Ampere's law: $\boxed{\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{encl}}}$ $\Rightarrow$ use if there are symmetries: plane (planar) $\sim$ const; $B \parallel$ to the plane
  - circumferential: straight wire $\frac{\mu_0 I}{2\pi r} \sim \frac{1}{r}$; toroid $\frac{\mu_0 NI}{2\pi r} \sim \frac{1}{r}$ (in) and 0 (out);
  - solenoid $\sim$ const (in) and 0 (out) $\to \vec{B} = \mu_0 nI\hat{z}$ where $N = \#$ turns and I is calculated per unit length
- when symmetries cannot help: $\vec{B}(r) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{r'}}{r^2}$; r where field is evaluated; r' vector from line element to r
- Recall: usually $I \parallel d\vec{l}$ and $I \parallel \vec{A} \to$ BC$_s$: $\boxed{B^{\parallel}_{\text{out}} - B^{\parallel}_{\text{in}} = \mu_0 \vec{K} \times \hat{n}; B^{\perp}_{\text{out}} - B^{\perp}_{\text{in}} = 0}$
- *cyclotron* motion: $qvB = \frac{mv^2}{R} \to \omega = \frac{qB}{m}$ (freq.); $R = \frac{mv}{qB}$ (radius)
- *diamagnetic* materials have lower $\vec{B}$ which does not change direction (opposite of *ferromagnetic*)
- Superconductors: outside surface $\vec{B}^{\perp} = 0$ ($\vec{B}$ only *tangential*!)
  - when cooled below certain temperature, materials have 0 resistance $\to$ explained by presence of *cooper* pairs: 2 electrons weekly bound with energy below Fermi energy s.t. favorable to pair up

2.3 Electrodynamics

- $\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \Rightarrow \oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{\partial t} = \mathscr{E}$ (electromotive force): *minus* sign because by energy conservation *induced* currents must oppose magnetic flux (currents always *reduced* by induced $\mathscr{E}$!)
- $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow \oint_S \vec{B} \cdot d\vec{a} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t}$ (*e.g.* charging capacitor involves *displacement* current)
- Recall when computing fluxes always use *cross-sectional* area $\Rightarrow$ multiply by N turns when necessary
- *mutual* inductance M: $\Phi_1 = M_{1,2} I_2$; $\Phi_2 = M_{1,2} I_1$; $\Rightarrow \Phi_2/I_1 = \Phi_1/I_2$ (depends purely on *geometry*)
- *self* inductance L: $\Phi = LI$; $\mathscr{E} = -L\frac{dI}{dt}$ (depends on *geometry*) $\Rightarrow$ to evaluate: compute flux; plug in Lenz's law
  as capacitors, inductors store energy $U_L = \frac{1}{2}LI^2 \Rightarrow$ for *solenoid* $\boxed{L/l = \mu_0 N^2 A}$
- Recall: if a wire is being *wound* around: magnetic flux is changing!
- *electric* dipoles: $\vec{p} = \sum_i q_i \vec{d}_i = \int \vec{r}\rho(r) d^3\vec{r} \Rightarrow V(r) = \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} \sim \frac{1}{r^2}$ hence $\vec{E} \sim \frac{1}{r^3}$
  tend to *align* with $\vec{E}$ field: $\vec{N} = \vec{p} \times \vec{E}$ (torque); $U = -\vec{p} \cdot \vec{E}$ (where $F = -\nabla U$)
- *magnetic* dipoles $\vec{m} = I\vec{A}$ (vector pointing *normal* to surface) $\Rightarrow \vec{B} = \frac{m_0}{4\pi r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) \sim \frac{1}{r^3}$
  $\vec{N} = \vec{m} \times \vec{B}$ (torque); $U = -\vec{m} \cdot \vec{B}$ (where $F = -\nabla U$) $\Rightarrow$ field *far* from current loop equals that of a *dipole*
- *Multipole* expansion: the n^{th} term $\propto 1/r^{n+1}$ for $n = \{0, 1, 2, ...\}$
  - in $\vec{E}$ field with net charge: *monopole* term dominates; in $\vec{B}$ field/ in $\vec{E}$ with $Q_{\text{tot}} = 0$ *dipole* term dominates

- in matter *polarization* **P** (electric dipole moment per unit vol.) causes *bound* charges: $\sigma_b = \mathbf{P}\cdot\hat{n}$; $\rho_b = -\nabla\cdot\mathbf{P}$ $\rightarrow$ to compute field just apply usual Gauss's law and other rules using these new charges
  - P $= \epsilon_0\chi_e\vec{E}$; $\vec{D} = \epsilon\vec{E}$ with $\epsilon = \epsilon_0(1+\chi_e) = \epsilon_0\kappa \Rightarrow$ new BC$_s$: $\boxed{\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f}$
  - For simplest configurations $\vec{E}$ transform by $\epsilon_0 \rightarrow \epsilon$: parallel plate capacitor $C = \frac{\epsilon A}{d} = \kappa\frac{\epsilon_0 A}{d}$
  - if *dielectric* change $\perp \vec{E}$: $\vec{D}$ uniquely determined - $\vec{E}$ changes; $\parallel \vec{E}$: $\vec{E}$ uniquely determined - $\vec{D}$ changes
- in matter *magnetization* **M** (magnetic dipole moment per unit vol.) causes *bound* current: $\vec{K}_b = \mathbf{M}\times\hat{n}$; $\vec{J}_b = \nabla\times\mathbf{M} \rightarrow$ to compute field just apply usual Ampere's law and other rules using these new currents
  $\mathbf{M} = \frac{\chi_m}{\mu}\vec{B}$; $\vec{H} = \frac{\vec{B}}{\mu}$
- Recall: if $\vec{E} \parallel \vec{B}$ since $\vec{v} \parallel \vec{E}$ then $\vec{v} \parallel \vec{B}$ and hence $\vec{F}_{\text{mag}} = 0$
- Recall: if charge *oscillates* back and forth, the field should be still maximized *near* particle

2.4 E-M waves & Radiation

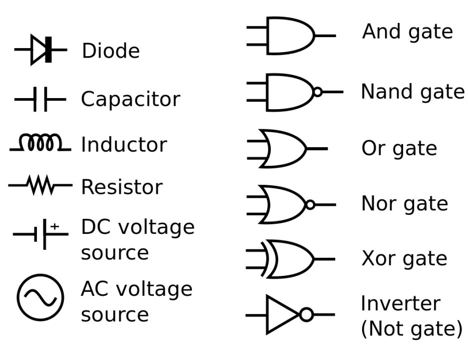
- in vacuum plane waves with speed $c = 1/\sqrt{\epsilon_0\mu_0}$: $\tilde{E}(\vec{r}) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-wt)}\hat{n}$; $\tilde{B}(\vec{r}) = (\hat{k}\times\tilde{E})/c$
- the *pointing* vector: $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \rightarrow$ energy per unit area, per unit time/Power per unit area
  - $P = \oint_S \vec{S}\cdot\vec{a}$ (Power); $I = \langle S\rangle = \frac{1}{2}c\epsilon_0 E_0^2$ (intensity)
- BC$_s$ at x-z plane: (i) $\epsilon_1(\tilde{E}_{0,I}+\tilde{E}_{0,R})_z = \epsilon_2(\tilde{E}_{0,T})_z$; (ii) $(\tilde{B}_{0,I}+\tilde{B}_{0,R})_z = (\tilde{B}_{0,T})_z$;
  (iii) $(\tilde{E}_{0,I}+\tilde{E}_{0,R})_{xy} = (\tilde{E}_{0,T})_{xy}$; (iv) $\frac{1}{\mu_1}(\tilde{B}_{0,I}+\tilde{B}_{0,R})_{xy} = \frac{1}{\mu_2}(\tilde{B}_{0,T})_{xy}$
- B_R reflected always *opposite* sign w.r.t. $B_T \rightarrow$ change also sign of k if wave goes opposite way
- perfect *conductor*: $E_T = 0 \rightarrow E_{0,I} = -E_{0,R}$ (all fields to the left cancel; B fields same direction, sum up)
- *accelerating* electric charge (for $v \lll c$) radiates s.t. $\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c} \propto q^2 a^2}$; a could depend on m (since $a = \frac{F}{m}$): $P \propto \frac{1}{m^2}$
- *oscillating* dipole with moment $p = p_0\cos(\omega t)$ radiates s.t. $\langle S\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{12\pi c}\right)\frac{\sin^2\theta}{r^2} \propto \frac{p_0^2\omega^2}{r^2}\sin^2\theta \rightarrow$ no radiation along dipole *axis* (recall: *monopoles* do not radiate!)
  - $\langle P\rangle_E = \frac{\mu_0 p_o^2\omega^4}{12\pi c} \propto p_0^2\omega^4$; $\langle P\rangle_B = \frac{\mu_0 m_o^2\omega^4}{12\pi c^3} \propto \frac{m_0^2\omega^4}{c^3} \lll \langle P\rangle_E$ (electric radiation *dominates*)
  - a sphere of total charge Q that expands and contracts has total radiated power equal to *zero*

2.5 Circuits

- $V_R = IR$; $V_C = \frac{Q}{C}$; $V_L = -L\frac{dI}{dt} \rightarrow$ energy stored in capacitors ($U_C = \frac{1}{2}CV^2$) and inductors ($U_L = \frac{1}{2}LI^2$); dissipated in resistors ($P_{\text{dis}} = IV = I^2R$) where $\boxed{R = \rho l/A}$ $\Rightarrow$ to relate it to force recall $\vec{F}\cdot\vec{v} = P$
- if you have n resistors with *equal* $R \Rightarrow$ in *series*: voltage *multiplicator* by n; in *parallel*: voltage *divider* by n
- *loop* rule: $\sum_i V_i = 0$; *junction* rule: $I_{\text{in}} = I_{\text{out}} \Rightarrow$ Thevenin equivalent: any combination of voltage source/ currents/ resistors is equivalent to 1 voltage source + 1 resistor
- For AC circuits use *Impedance*: $Z_R = R$; $Z_C = \frac{1}{i\omega C}$; $Z_L = i\omega L$ with $Z_{\text{tot}} = |\sum_i Z_i|$
  - RC and RL circ.: $\tau_{RC} = RC$ (discharging const.); $\tau_{RL} = \frac{L}{R}$ (response time) $\Rightarrow$ time to drop V by $1/e$
  - RLC circ.: $\omega_{\text{res}} = \frac{1}{\sqrt{LC}}$ where $\langle P\rangle = I_{\text{rms}}^2 R$ with $I_{\text{rms}} = \frac{\sqrt{2}}{2}I$ (like damped. harmonic oscillator)
  - LC circ.: $\omega = \frac{1}{\sqrt{LC}}$ (like simple harmonic oscillator)
  - *resonance*: frequency where *imaginary* part of impedance Z goes to zero!
  - for $\omega \rightarrow \infty$ capacitors act like *short* circuit; inductors like *open* circuit; for $\omega \rightarrow 0$ *viceversa*

- LP-filter: RC circ. with C on output (capacitor is *low*/ connected to the ground) $\left|\frac{V_0}{V_i}\right| = 1/\sqrt{1+(RC\omega)^2}$
- HP-filter: RC circ. with R on output (resistor is *low*/ connected to the ground) $\left|\frac{V_0}{V_i}\right| = RC\omega/\sqrt{1+(RC\omega)^2}$;

  RL circ. with L on output (inductor is *low*/ connected to the ground) $\left|\frac{V_0}{V_i}\right| = R/[\omega L\sqrt{1+(R/\omega L)^2}]$; generally for filters: compute impedance and check limit cases $\omega \to \{0; +\infty\}$
- sudden switch can be thought as *ultra-high* frequency event ($\omega = \infty$ at $t = 0$) which gradually relax to small frequencies s.t. $\omega = 0$ at $t = \infty \Rightarrow$ when switch is closed V_L is *max* since it's $\propto \frac{dI}{dt}$
- Recall: never forget *internal* resistance, if it's mentioned it's important!
- to maximize power transmitted one needs *impedance* of source to be *equal* to that of output
- OP-AMP: gain $\uparrow$; input impedance $\uparrow$; output impedance $\downarrow \Rightarrow$ use feedback circuit to control gain
- *transformers* consist of two coils with $V_s/N_s = V_p/N_p$, hence by energy cons. $I_pV_p = I_sV_s \Rightarrow I_p = I_sN_s/N_p$
- The hall effect: used to determine *sign* of charge carriers according to $R_H = -\frac{1}{nec}$
- Logic gates: elements take on *discrete* values $\Rightarrow$ AND: *true* only if both A and B are true
  - OR: always *true* except if both A and B are false; NOT: returns opposite of A ($\bar{A}$)
  - NAND; NOR: just usual AND; OR for *inverted* inputs $\Rightarrow \overline{A\cdot B} = \bar{A}+\bar{B}$; $\overline{A+B} = \bar{A}\cdot\bar{B}$
  - a series of NAND or any other logical gate can be combined to create any sequence of logical gates



3 Waves

3.1 Foundations

- $\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2} \to$ for any f the related function $f(x \pm vt)$ solves equation
  - *linear* solutions: if f, g solves equation also $Af + Bg$ does!
  - wave travelling to the *right*: $-$ sign $(x - vt)$; *left*: $+$ sign $(x + vt)$
  - *standing* wave: $f(x,t) = A(x)B(t) \to$ shape oscillates in time but doesn't go anywhere; can be rewritten as sum of *left* and *right* moving wave
  - Intensity: $I \sim A^2$ (energy carried); $\lambda = 2\pi/k$; $\to$ de-coupled waves add their intensities *separately*
  - $T = 2\pi/\omega$; $\omega = 2\pi f \Rightarrow k$ is wave-vector whose direction tells us where the wave propagates
- dispersion relation is $\omega(k)$: *phase* velocity $v_p = \frac{w}{k}$ (velocity of individual crest)
  - *group* velocity $v_g = \frac{d\omega}{dk}$ (speed of wave packet/ at which information travel; must be *less* than c)
    v_p can be greater than v_g and even than c!
  - classically $\omega(k) = vk$; quantum particles $\omega(k) = \frac{\hbar k^2}{2m}$
- wave examples: *string*: $v = \sqrt{\frac{T}{\mu}}$ with T tension and μ mass density; *sound* $c_s = \frac{\kappa}{\rho}$ with κ measure of stiffness and ρ as density: ratio of change in pressure to fractional volume compressed
- in medium with index of refraction n: $v_1 = v_0/n$; $\lambda_1 = \lambda_0/n$; $f_1 = f_0$ (freq. is *const*)
  minimum speed of particle in medium is just $v = v_i/n$
- Polarization gives direction of wave $\Rightarrow$ *longitudinally* pol. wave: *same* direction as displacement of wave medium
  - polarized in direction $\hat{n}_0$: $I = I\cos^2\theta$ with $\hat{n} \cdot \hat{n}_0 = \cos\theta$
  - two polarizers at $\frac{\pi}{2}$ no light transmitted; two at $\frac{\pi}{2}$ and one in the middle at $\frac{\pi}{4}$: output is *not* zero!
  - if *unpolarized* (light in every direction) light shines on polarizer: $I = \langle I \rangle = I_0/2$
  - Brewster angle $\theta_B = \arctan\left(\frac{n_1}{n_2}\right)$: light reflected off polarized $\perp$ to plane formed by incident ray and normal to surface $\Rightarrow$ if light polarized $\parallel$ to incident plane: *no reflection* at all
- Doppler effect: $\boxed{f = \left(\frac{v + v_r}{v - v_s}\right) f_0}$ where v_r of *receiver* and v_s of *source*
  - if source moving *away*: v_s negative, $f \downarrow$; *towards*: v_s positive, $f \uparrow$
  - formula is only valid if receiver and source moving *directly* towards or away from each other
  - f is *constant*: the falling freq. sound only given by varying angle
  - Be very careful with velocity of receiver and source: do not confuse with velocity of *medium* v
- Pipes: *open* end is a node (no change in pressure!); *closed* end is anti-node
  - *lowest* mode approach: open pipe $\to \frac{\lambda}{2}$ so $\lambda_{\max} = 2L$; closed pipe $\to \frac{\lambda}{4}$ so $\lambda_{\max} = 4L$
    $\to$ *fundamental* harmonics are respectively $f_m = \{\frac{mv}{2L}, \frac{mv}{4L}\}$
  - $f_{\text{beat}} = f_1 - f_2 \to$ if I choose f_0 to be fundamental harmonics then the n^{th} harmonic has frequency $f_n = nf_0$
- wavelengths orders of magnitude: *radio* (mm to km); *visible* (400nm to 700nm); *x-rays* (0.01nm to 10nm)

3.2 Interference patterns

- general interference pattern: $\Delta\delta = 2m\pi$ (*constructive*); $\Delta\delta = (2m+1)\pi$ (*destructive*) where $\Delta\delta = k\Delta x$
- Double slit (separation d): $\boxed{d\sin\theta = m\lambda}$ (*constructive*); $\boxed{d\sin\theta = (m + 1/2)\lambda}$ (*destructive*)
  \# of fringes given angular aperture $\alpha = 2\theta \Rightarrow \# = 2m$!
- Single slit (large a): $\boxed{a\sin\theta = m\lambda}$ (*minima*) $\Rightarrow$ first minima gives *width* of central maximum: $2L\tan\theta \approx 2L\theta$
  to find *sharpest* image minimize first diffraction pattern according to eq. $\sin\theta \approx \theta = \frac{\lambda}{d}$

- Optical path length ⇒ wave travels different distances in different media: $\Delta x = nd$ where $\Delta\delta = k\Delta x$
  - for $n \to \infty$ slows down so much that goes over infinitely many cycles
  - in thin film of thickness d there are two sources of phase-shift:
    (1) going from medium n_1 to n_2: $\Delta\delta = \{0 \text{ if } n_2 < n_1 \text{ },\pi \text{ if } n_2 > n_1\}$ (corresponds to $\Delta\lambda = \{0, \frac{\lambda}{2}\}$)
    (2) $\Delta x = 2dn_2$ (path length) ⇒ in tot. if $n_2 > n_1$ *constructive* inter.: $\boxed{2dn_2 = (m + 1/2)\lambda}$

- Bragg diffr.: $d \sin\theta = \frac{n\lambda}{2}$ (*constructive*) → from crystal lattice modelled as set of ∥ planes at distance d apart

- given number of slits per unit length *constructive* interference appear at $\frac{L}{N}\sin\theta = m\lambda$

- Rayleigh criterion for *circular* apertures: first diffraction minimum at $\boxed{D\sin\theta = 1.22\lambda}$
  - minimum angle for two images to be resolved: $\theta \approx \frac{1.22\lambda}{D}$ → if they give you the frequency, recall that to get the wavelength it's simply $\lambda f = c$
  - Rayleigh scattering (for $\lambda \ggg a$): $I \propto I_0\lambda^{-4}a^6$ ⇒ think of this formula when particle scattering is mentioned

- Interferometer: a fringe shift occurs every $\frac{d}{\lambda}$ hence # of fringes is $m = \frac{2d}{\lambda}$

- interference is produced if sources are *coherent*: 500Hz already much greater than max-freq. of human eye

- Resolving power of spectrometer is $\Delta\lambda/\lambda$

3.3 Optics

- Geometric optics (for $\lambda \lll a$): $n_1 \sin\theta_I = n_2 \sin\theta_T$; $\theta_I = \theta_R$
  total *internal* reflection is when $\sin\theta_T > 1$ or $\frac{n_1}{n_2}\sin\theta_I > 1$

- be always careful: angles w.r.t. the *horizon* are different than angles w.r.t. *normal*

- *plane* mirrors: $p = -i$ with p object; i image → if $i < 0$ image to the *left* of mirror

- *spherical* mirrors: $\boxed{\frac{1}{p} + \frac{1}{i} = \frac{1}{f}}$ with f as the distance to focus where all *parallel* rays converge
  - $m = -\frac{i}{p}$ (magnification) → sing determines orientation: + *upright*; − *inverted*
  - for *idealized* spheres $f = \frac{R}{2}$ (+ if center of curvature on *same* side of incoming light; − viceversa)
  - to draw image: 1 light ray ∥ and 1 going through *focus*

- Lenses are *converging*: 2 convex surfaces, $f+$; *diverging*: 2 concave surfaces, $f-$
  - $\boxed{\frac{1}{p} + \frac{1}{i} = \frac{1}{f}}$ → Recall: distances are + when on the other side of lens and − if they return back
  - in terms of radii of of curvature of two surfaces of lens: $f = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$
  - when you have multiple lenses treat them *independently*
  - magnification of telescope made of *objective* and *eye-piece* is $M = \frac{f_o}{f_e}$
  - to draw image for *converging* lens: rays through *both* focuses; *diverging* lens: 1 ray through focus and 1 through center ⇒ in general when rays *converge* image is *real*; when they *diverge* is *virtual*!

4 Thermo & Stat-Mech

4.1 Microscopic Ensembles

- Canonical distribution → an ensemble in contact with *heat* reservoir (T,V and # particles are *fixed*)
  - $P_r = \frac{e^{-\beta E_r}}{Z}$ where $Z = \sum_r e^{-\beta E_r}$ (partition function); $\beta = \frac{1}{k_B T}$
  - $\bar{y} = \frac{1}{Z}\sum_r y_r e^{-\beta E_r} \to \bar{E} = -\frac{\partial \ln Z}{\partial \beta}$; $\bar{\Delta E}^2 = -\frac{\partial \bar{E}}{\partial \beta} = -\frac{\partial^2 \ln Z}{\partial \beta^2}$
  - $dW = \bar{X}dx$ where $\bar{X} = -\frac{\partial \bar{E}}{\partial x} = \frac{1}{\beta}\frac{\partial Z}{\partial x}$ → implies $\bar{p} = P = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}$
  - If there are N subsystems $Z_{\text{tot}} = \Pi_i Z_i$ → if systems of *indistinguishable* particles $Z_{\text{tot}} = \frac{1}{N!}\Pi_i Z_i$
  - I.M.G.: $Z_i = \frac{V_0}{\hbar^3}\int_{-\infty}^{\infty} e^{-\frac{\beta p^2}{2m}} d^3p = V_0\left(\frac{2\pi m}{\hbar\beta}\right)^{\frac{3}{2}} \propto \beta^{-\frac{3}{2}}$
  - relativistic particles: $Z_i = \frac{V_0}{\hbar^3}\int_{-\infty}^{\infty} e^{-\beta pc} 4\pi p^2 dp = \frac{8\pi V_0}{(\hbar\beta c)^3} \propto \beta^{-3}$
- microcanonical ensemple: *fixed* energy and temperature
- if we allow particles to be exchanged: $\mathscr{Z} = \sum_{\text{APS}} e^{-\beta E_r - \alpha N_r}$ (gran partition function); $\alpha = -\mu\beta$
  - μ is the chemical potential (if $\mu_A = \mu_B$ *no particle* flow) → ; $\bar{N} = -\frac{\partial \ln \mathscr{Z}}{\partial \alpha} = \frac{1}{\beta}\frac{\partial \ln \mathscr{Z}}{\partial \mu}$
- Entropy is a measure of uncertainty on state of system: $S = k_B \ln \Omega$ where Ω = # of microstates
  - $S = -k_B \sum_i p_i \ln p_i = \frac{\partial}{\partial T}(k_B T \ln Z)$
  - at fixed temperature: $S = k_B(\ln Z + \beta \bar{E})$; for I.M.G.: $S = Nk_B \ln\left(\frac{VT^{\frac{3}{2}}}{N}\right)$ + const
  - if types of particles are the same *sum up* individual entropies otherwise use the usual formula $S = k_B \ln \Omega$
- Recall: $\ln(n!) \approx n(\ln(n) - 1)$; $\left(\begin{array}{c} N \\ M \end{array}\right) = \frac{N!}{M!(N-M)!}$ (ways of separating a group of M people from a pool of N)

4.2 Thermodynamics

- Equipartition theorem: each d.o.f. (quadratic term) in Hamiltonian contributes $\frac{1}{2}k_B T$ to internal energy
  # of *transitional* d.o.f.: # of dimensions;
  # of *rotational* d.o.f.: # of dimensions -1 (linear mol.); # of dimensions (non-linear mol.)
  # of *vibrational* d.o.f.: $3N - 6$ for $N > 2$ (linear mol.) → the only to depend on the # of particles
  - vibrational energies of diatomic molecule are approximately those of harmonic oscillator ($k_B T \approx \hbar\omega = hf$) → these modes only become *frozen* at temperatures 1 order of magnitude larger than room-temp ($\sim 10^3$K)
  - generally the higher T the more modes are unfrozen
    for *low* T: only transitional (atoms as *rigid dumbells*)
    rotational freeze at $T \sim 1$K (they are *free* at room temp.)
    for $T \ggg 1$ all modes unlock (atoms become *springy*)
- Laws of thermodynamics: at equilibrium $T_A = T_B$ (most probable system) with $T = 1/\frac{\delta S}{\delta E}$ → S is maximized
  - $\Delta U = Q - W \to dE = dQ - dW$ where $dQ = TdS$; $W = \bar{X}_\alpha dx_\alpha = PdV$
  - $\Delta S \geq \int \frac{dQ}{T}$ where $\frac{dQ}{T} = S_{\text{rev}}$ (reversible process) → $\Delta S_{\text{univ}} \geq 0$
  - for reversible processes $\Delta S_{\text{univ}} = 0$ → note on single interactions we *can* have $\Delta S < 0$.
  - Quasi-static (QS) processes correspond to infinitely slowly as a succession of thermodynamics equilibriums
  - $S(T \to 0) = 0$ (*not always* if ground state is degenerate)
  - A cooler body can *never* just heat up a hotter body
- Heat capacities: $c_y = T\left(\frac{dS}{dT}\right)_y = \left(\frac{dQ}{dT}\right)_y$ hence → $c_v = \left(\frac{dQ}{dT}\right)_V = \left(\frac{dE}{dT}\right)_V$; $c_p = \left(\frac{dQ}{dT}\right)_P$
  - for conductors (*metals*) $c_v \propto T^3$ → if material is superconductor c_v jumps up and then goes back down
  - for relativistic gases $c_v = 3k_B T$

- Recall: spin always plays a role in determining the specific heat of an object

- Enthalpy: $H = E + PV$ ($dH = TdS + VdP$); Free energy: $F = E - TS$ ($dF = -SdT - PdV$);
  Gibbs free energy: $G = F + PV$ ($dG = -SdT + VdP$)
  - Maxwell relations can be found by equating the *second* derivative of each potential in terms of their parameters (think of the parameters as partial derivatives of one of the functions)
  - when # of particles is *not* fixed, chemical potential becomes useful $\mu = \left(\frac{\partial F}{\partial N}\right)_{T;V} = \left(\frac{\partial E}{\partial N}\right)_{S;V}$
  - $T = \left(\frac{\partial U}{\partial S}\right)_V$; $P = \left(\frac{\partial U}{\partial V}\right)_S \to \left(\frac{\partial P}{\partial S}\right)_V = -\left(\frac{\partial T}{\partial V}\right)_S$

- Ideal Gases: $\Omega(E,V) = BV^N E^{\frac{3N}{2}}$; $\bar{p}V = PV = nRT = Nk_BT$
  - $E = \frac{3}{2}Nk_BT$; $c_p = c_v + Nk_B$; $c_v = \frac{\#\text{d.o.f.}}{2}Nk_B$; $\gamma = \frac{c_v}{c_p} \to \gamma = \frac{5}{3}$ (mono-atomic); $\gamma = \frac{7}{5}$ (di-atomic);
  - Isothermal ($T = $ const): $\Delta E = 0$; $\Delta Q = \Delta W = nRT\ln\frac{V_f}{V_i}$; $\Delta S = nR\ln\frac{V_f}{V_i} \to$ during isothermal exp. $F \downarrow$
  - Isocoric ($V = $ const):$\Delta E = \Delta Q = c_v\Delta T$; $\Delta W = 0$; $\Delta S = c_v \ln\frac{T_f}{T_i}$
  - Isobaric ($P = $ const):$\Delta E = c_v\Delta T$; $\Delta Q = c_p\Delta T$; $\Delta W = P\Delta V$; $\Delta S = c_p \ln\frac{T_f}{T_i}$
  - Adiabatic-isentropic ($Q_{\text{in}} = Q_{\text{out}} = 0$): $\Delta E = -\Delta W = c_v\Delta T$; $\Delta S = \Delta Q = 0$
    $PV^\gamma = $ const; $V^{\gamma-1}T = $ const
  - given same fractional increases $\Delta S_p > \Delta S_v > \Delta S_T > \Delta S_{\text{adiabatic}}$
  - we can also write $\Delta Q = cm\Delta T$ where c is the specific heat of a material ($c_{\text{water}} = 418\text{JK}^{-1}\text{g}^{-1}$)

- Free Expansion: $Q = 0$ (system adiabatically isolated); $W = 0$ (no work in the process) $\to \Delta E = 0$
  - if I.G. since $E \propto T$: $\Delta T = 0$; if not I.G. for $V_2 > V_1$: $T_2 < T_1$ (temperature decreases)
  - This is not QS/reversible process $\Delta S \neq 0 = nR\ln\frac{V_f}{V_i} \to \Delta S_{\text{FE}} = \Delta S_T$ so it corresponds to *minimum* entropy change for expansion (recall adiabatic does *not* mean $\Delta S = 0$)

- Heat engines: $\Delta E_{\text{tot}} = 0$ (cycle); $\Delta W = \Delta Q_{\text{in}} - \Delta Q_{\text{out}} = \int TdS$
  - $\eta = \frac{\Delta W}{\Delta Q_{\text{in}}} = 1 - \frac{\Delta Q_{\text{out}}}{\Delta Q_{\text{in}}}$ (efficency)
  - for reversible processes $\Delta S_{\text{universe}} = 0$ so $\Delta S_{\text{machine}} = \frac{\Delta Q}{T}$ which implies $\eta = 1 - \frac{T_{\text{out}}}{T_{\text{in}}}$
  - Cornout Cycle: 2 adiabatic; 2 isothermal (rectangle in S-T space)
  - *clockwise* paths (expansions) in P-V;S-T planes do *positive* work

- Van der Walls gases: $\left(P + \frac{N^2a}{V^2}\right)(V - Nb) = Nk_BT \to a$, b respectively measure *attraction*, *size* of particles

- Recall: if gases are identical and one removes partition *nothing* changes $\to \Delta S = 0$ (no additional states)

- *Never* assume the gas is monotone unless it explicitly says it!

4.3 Quantum Statistics

- Average energy: $\langle\epsilon\rangle = \int_0^\infty \epsilon\bar{n}(\epsilon)\rho_\epsilon d\epsilon$; Average # of particles: $\langle N\rangle = \int_0^\infty \bar{n}(\epsilon)\rho_\epsilon d\epsilon$

- to derive average occupation number at energy level r, *fix* energy and think in terms of number of particles

- ρ is the density of states with $\rho_k = \frac{gVk^2}{2\pi^2}$ and $g = \#$ degeneracies (careful with what this number is)
  $\to \rho_\epsilon = \rho_k\left(\frac{d\epsilon}{dk}\right)^{-1}$ with $\epsilon = pc = \hbar ck$ (relativistic) and $\epsilon = \frac{\hbar^2k^2}{2m}$ (classical)

- Bosons are *indistinguishable* and as many as you want in 1 state
  - at energy ϵ_r: $\mathscr{Z}(\epsilon_r) = \sum_i e^{-\beta(\epsilon_r-\mu)\cdot i)}$ with $i = \#$ particles; occupation $\boxed{\bar{n}^{\text{BE}}(\epsilon_r) = 1/(e^{\beta(\epsilon_r-\mu)} - 1)}$
  - for $T \to 0$ collection of bosons in ground state approaches infinity

- Fermions are *indistinguishable* and at most 1 per state

  - at energy ϵ_r: $\mathscr{Z}(\epsilon_r) = 1 + e^{-\beta(\epsilon_r - \mu)\cdot)}$; occupation $\boxed{\bar{n}^{\mathrm{FD}}(\epsilon_r) = 1/(e^{\beta(\epsilon_r-\mu)} + 1)}$
  - for $T \to 0$ $\bar{n}^{\mathrm{FD}}(\epsilon_r) = \{0 \text{ if } \epsilon_r > \mu; 1 \text{ if } \epsilon_r < \mu; \frac{1}{2} \text{ if } \epsilon_r = \mu = \epsilon_F \text{ (Fermi energy)}\}$
  - when there are only two states think of Fermi Dirac statistics

- free electrons only behave kinematically and by Pauli exclusion principle: $E_F = \frac{\hbar^2 k_F^2}{2m}$ with $k_F = (3\rho^2\pi^2)^{\frac{1}{3}}$

  $\rho =$ electron density; $n = \frac{N}{V} \to \boxed{k_F \propto n^{\frac{1}{3}};\ E_F \propto n^{\frac{2}{3}}}$

- velocity in materials is always $v_{\text{ind}} = \sqrt{\frac{\alpha RT}{m}}$: in I.G. for ind $= \{\text{rms}, \text{mode}, \text{avg}\}$ $\alpha = \{3, 2, 8\}$
- Recall: if Maxwell Boltzmann description $\vec{v}_{\text{avg}} = 0$ (including direction)
- Debeye & Einstein models assumed $3N$ *oscillator model*$\to$ Einstein: all with *same* frequency; Debeye: *spectrum* of frequencies
- Power emitted by blackbody is $P = \sigma\epsilon A T^4 \propto T^4$
  - $I = \frac{2\hbar\omega^3}{c^2} \cdot \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \propto \omega^3$ (intensity)
  - The peak of the spectrum is at $\boxed{\lambda_{\text{max}} = 2.9 \cdot 10^{-3}\text{K} \cdot \text{m}/T \propto 1/T}$

5 Quantum Mechanics

5.1 Foundations

- $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$ → separation of variables: t) $\frac{\partial\psi}{\partial t} = -i\frac{E}{\hbar}\psi$; x) $H\psi = E\psi$ (T.I.S.E.)
  - general solution $\Psi(x,t) = \sum_n c_n\psi_n e^{-E_n t/\hbar}$ where $c_n = \langle\psi_n|\psi\rangle$ and $P_n = |c_n|^2$
    $\langle\psi_n|\psi_m\rangle = \delta_{mn}$ (eigenfunctions are orthogonal)
  - the observed quantities are the eigenvalues q_n; while the expected value $\langle Q\rangle = \sum_n q_n|c_n|^2$
  - Hamiltonian eigenstates are *stationary*: expectation values are *constant*: $\frac{d\langle\hat{Q}\rangle}{dt} = 0$; $[\hat{Q}; H] = 0$
    → if $\Psi(x,0) = \Psi_n(x)$ (initial state= stationary); the probability of getting E_n at any other time is 1
  - *superposition* of eigenstates are not stationary and introduce a factor $\propto \sin\theta/\cos\theta$ where $\theta \propto (E_1 - E_2)t/\hbar$
    → Recall:the energy eigenvalues and relative probabilities are still *constant* in time
  - Ψ needs to be normalizable s.t. $\int_{-\infty}^{\infty}|\Psi||^2dx = 1$ → recall Ψ is just a wavefunction; $|\Psi|^2$ is the probability distribution (*e.g.* $\int_a^b|\Psi|^2dx$ is probability to find particle in $x \in (a,b)$)
  - if problem doesn't explicitly state Ψ is *normalized*, you should do it yourself before computing anything else
  - Ψ has dimensions $d/2$ where $d = \#$ spatial dimensions
- Hermitian operators: $\langle f|\hat{Q}|f\rangle = \langle\hat{Q}f|f\rangle$ as $\hat{Q} = \hat{Q}^\dagger$ → Hermitian conjugate is transpose + conjugate: $A^\dagger = (A^T)^*$
  - $\hat{Q}\psi_n = q\psi_n$ where q is *real* (eigenvalues *must* be real and represent *observables*)
  - any operator involving 1 derivative *without* the factor of i, it cannot be hermitian
  - total energy operator is $E = i\hbar\frac{\partial}{\partial t}$
- Recall: the expect. value of an imaginary number is *zero* (not observable)→ if Ψ real and $\hat{Q}$ imaginary: $\langle\hat{Q}\rangle = 0$
- Commutator $[A,B] = \mathrm{AB} - \mathrm{BA}$ → when evaluating them always apply them to a *wavefunction*
  - $[\mathrm{AB}, C] = A[B,C] - [A,C]B$; $[A;B] = -[B;A]$
  - if commutator is *zero*, operators are compatible and constitute complete set of *simultaneous* eigenfunctions ψ_n
  - if operator $\hat{O}$ commutes with Hamiltonian, the corresponding observable is *conserved*
  - uncertainty principle: $\sigma_x\sigma_p \geq \frac{\hbar}{2}$ since $[x,p] = i\hbar$ where $p = i\hbar\frac{\partial}{\partial x}$ (*minimum* corresponds to Gaussian wave-packet)
  - $\sigma_t\sigma_E \geq \frac{\hbar}{2}$ → to make computations approximate $\Delta x\Delta p \approx \hbar$; $\Delta t\Delta E \approx \hbar$
  - when they ask you for a *lower* bound (like minimum radius) think of *uncertainty* principle
- the wavefunction ψ is always *continuous*; $\frac{\partial\psi}{\partial x}$ is only discontinuous where $V(x) \to \pm\infty$
  - ψ_n has n nodes, so ψ_0 (ground state) has no nodes (*no* points where particle is guaranteed not to be found)
  - if they give you a wavefunction and ask for its respective potential compute $\frac{\partial^2\psi}{\partial x^2}$ and compare it to T.I.S.E.
  - Always determine if ψ should be oscillating or decaying by looking at T.I.S.E. → recall it's $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ so if $E - V > 0$ *oscillating*; if $E - V < 0$ *decaying*
  - if $V(x)$ is even: ψ can be either even or odd → parity alternates so that ψ_0 is even; ψ_1 is odd, etc...
    if $\psi(x) = \psi(-x) \to \langle x\rangle = 0$
    if wavefunction is even: always node in the *middle*
- the energy of quantum system made of only a *rod* connecting two point masses is given by the *rotational* degrees of freedom s.t. : $T = L^2/2I = \hbar^2 n(n+1)/2I$ where $n = l$

5.2 1-particle systems

- S.H.O.: $E_n = \hbar\omega(n+\frac{1}{2})$; $\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0$ where $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$; $a|n\rangle = \sqrt{n}|n\rangle$
  - $\langle T\rangle = \langle V\rangle = E_n/2 \rightarrow$ more generally for $V(x) = \lambda x^n$: $\langle T\rangle/\langle V\rangle = n/2$
  - 3d: $\psi_N = \psi_{n,x}\psi_{n,y}\psi_{n,z}$ with $E_N = (n_x+n_y+n_z+\frac{3}{2})\hbar\omega = (N+\frac{3}{2})\hbar\omega$
  - if there is a wall on one side of S.H.O. potential all *even* states disappear
  - Recall: classical harmonic oscillator at ground state has energy *zero* (particle sitting at $x=0$)
- eigenstates of x: $\psi_a(x) = \delta(x-a)$; of p: $\psi_a(x) = \frac{1}{\hbar\sqrt{2\pi}}e^{\frac{iax}{\hbar}} = \delta(p-a) \rightarrow$ for a particle to have a definite position/momentum they have to be in the respective *eigenstates*!
- free particles: $\psi(x) = e^{\pm ikx}$; $E = \frac{\hbar^2k^2}{2m} = \hbar\omega$ with $\omega = \frac{\hbar k^2}{2m}$
  - can carry *any* positive energy: cannot exist in a *stationary* state and is *not* a normalizable solution
  - *normalized* wave-packet constructed by forming *continuous* superposition of $\psi_k(x)$ for different values of k
- δ-function potential ($V = -\alpha\delta(x)$): like free particles with BC$_s$: $\Psi(0^-) = \Psi(0^+)$; $\Delta\left(\frac{\partial\psi}{\partial x}\right)_{x=0} = -\frac{2m\alpha}{\hbar^2}\Psi(0)$
  - only 1 bound state with $E<0 \rightarrow$ if $V = \alpha\delta(x)$: only *scattering* states since by *tunneling* it will pass through the barrier if it eventually must come back
- finite square well: since V is *even* then ψ_0 is *even* $\rightarrow$ outside well *decaying* exponentials $\psi \propto e^{-kx}$; inside well *oscillating* solutions $\propto \sin, \cos$ (as well gets shallower, excited states disappear until there is only 1 bound state)
- particle in a box: free particle with 1 BC: $\psi(0) = \psi(L) = 0 \rightarrow \psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$; $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$
- Bound states: $v_{\min} < E < \min(V_{-\infty}, V_{+\infty}) \rightarrow$ *discrete* set of E_n and *normalizable* wavefunctions
- Scattering states: $E > \min(V_{-\infty}, V_{+\infty}) \rightarrow$ *continuous* set of E_n and *not normalizable* wavefunctions
  - if $E > \max(V_{-\infty}, V_{+\infty})$ 2 states per energy level, otherwise only 1 state per energy level
  - e^{ikx} for $k>0$: plane wave moving to the *right*
  - reflection coefficient $R = \frac{|B|^2}{|A|^2}$; transmission coefficient $T = \frac{k_R}{k_L}\frac{|C|^2}{|A|^2} \rightarrow R+T=1$
  - if $k_1 = k_2$ there is *no reflection*!; if $k_1 = 0 \vee k_2 = 0$ there is *no transmission*
  - when $E < V_{\max}$ particle can still *tunnel* and be on other side but it has to come back eventually ($R=1$)
  - given de Broglie wavelength $\lambda = \frac{\hbar}{p}$: $E = \frac{\hbar^2}{2m\lambda}$ with $\lambda = \frac{\hbar}{\sqrt{2mE}} \rightarrow$ for particles scattering think of $T = \frac{\hbar^2}{2m\lambda}$

5.3 Hydrogen atom & 3d-QM

- Bohr model: electrons in circular orbits with quantized values of angular momentum $L = n\hbar \rightarrow$ electrons in a given shell do not *radiate*
- with radial potential $V(r) \rightarrow \Psi = R(r)Y(\theta,\phi)$ where Y are spherical harmonics
  - angular momentum $L = \vec{r}\times\vec{p}$: $[L_x, L_y] = i\hbar L_z$ (with cyclic permutations)
  - $L_z = -i\hbar\frac{\partial}{\partial\phi} \rightarrow L^2\psi = \hbar^2 l(l+1)\psi$; $L_z\psi = \hbar m_l\psi$ with $m = \{-l,..,l\}$
  - if angular part of Ψ is equal to a spherical harmonics then Ψ has definite $L_z = m$ and $L_{\text{tot}} = l$
  - if $\psi \propto \cos(m\phi)$ possible eigenvalues are $\pm m\hbar$ as $\cos(m\phi) = \frac{e^{im\phi}+e^{-im\phi}}{2}$
  - L^2 commutes with all L_i
  - different coordinates commute with each other $\rightarrow [x,y] = [x,z] = [x,p_y] = ... = 0$!
- Hydrogen atom has $V = -\frac{e^2}{4\pi\epsilon_0 r}$ and $E_n = -\frac{\hbar^2}{2\mu a^2n^2}$ where $a = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$ is the *Bohr* radius
  - $E_n \propto \mu$ (reduced mass) $\rightarrow$ if we have *positron* instead of proton then $\mu' = \frac{\mu}{2}$ and hence $E_0' = \frac{E_0}{2}$
  - $E_n \propto 1/n^2$; $E_n \propto Z^2$ (# of protons); $E_n \propto (q_1q_2)^2$ (not $\propto$ (tot-charge)^4)

- the ground state energy $\boxed{E_0 = -13.6\text{eV}}$ $\rightarrow$ for hydrogen like atoms the *binding* energy is $E_B = Z^2 E_0$
- for each n: $l = \{0, 1, ..., n-1\}$ $\rightarrow$ ground state has *zero* angular momentum
- $l \leq k$ where k is the degree of polynomial $\rightarrow$ *odd* l for *odd* ψ in r (valid for l even as well)
- $l = 0$ when ψ is *symmetric* on every axis
- if two Ψ_s are *spherically* symmetric they have the same l
- for $l \neq 0$ $\Psi = 0$ at origin $\rightarrow$ states with $l = 0$ have higher probability to be found near the origin

- in transitions from n_f to n_i: $\Delta E = E_0(1/n_f^2 - 1/n_i^2)$; $\lambda = \frac{hc}{\Delta E}$; $f = \frac{E_0}{h}(1/n_f^2 - 1/n_i^2)$
  - if electron bombard from *outside* of atom $n_i \rightarrow \infty$
  - Lyman series: $n_f = 1$; Bolmer series $n_f = 2$ (when looking for longst wavelength take $n_i \rightarrow \infty$)
  - *Selection rules*: transition between states can only happen if:
    $\Delta m_l = \pm 1$ or 0; $\Delta l = \pm 1$ ($\neq 0$); $\Delta j \pm 1$ or 0; $\Delta m_s = 0$
    assume wavelength of electromagnetic radiation to be *large* compared to size of atom

- for a given n there are n^2 possible combination of l and m
  - $2n^2$ possible orbitals (to account for spin up and down); $2(2l+1)$ possible states in each orbital
  - shells fill in order from *smaller* values of l: $\{s, p, d\} = \{l = 0; l = 1; l = 2\}$
  - for a given spin: the *higher* L the smaller the energy $\rightarrow$ state with *highest* total spin has lowest energy

- *fine-structure* constant: defines strength of electromagnetic interaction $\alpha = \frac{\mu\hbar}{ac} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$
- ground state of Helium is *singlet*: spatially symmetric and antisymmetric in spin
- corrections to hydrogen energy in descendent order of magnitude:
  - *fine structure* ($\sim \alpha^2 E_0$): spin-orbit coupling breaks degeneracy in l but keeps that in m $\rightarrow$ like *Zeeman* effect for internal $\vec{B}$ where $\Delta H = \frac{e}{2m}(\vec{L} + 2\vec{S}) \cdot \vec{B}$ with $\frac{e}{2m}$ as the electron classic *gyromagnetic* ratio
  - *Lamb shift* ($\sim \alpha^3 E_0$): splits $2s$ and $2p$ with $j = 1/2$ $\rightarrow$ like *Stark* effect for internal $\vec{E}$ filed where $\Delta H = e\vec{E}\cdot\vec{r}$ (perturbation is *odd* so 1^{st} order effect on any *even* state is 0)
  - *hyperfine structure* ($\sim \frac{m_e}{m_p}\alpha^2 E_0$): spin-spin coupling is given by tendency of spins to anti-align to $\vec{B}$ field (energetically favorable) and splits ground state depending if spins are in *singlet* or *triplet* state $\rightarrow$ triplet needs more energy caused spins are *aligned*; in this transition the famous 21cm line is produced ($\sim 5{\cdot}10^{-6}$eV)

5.4 Spin

- intrinsic angular momentum of particle: $S_z\Psi = \hbar m_z \Psi$; $S^2\Psi = \hbar^2 s(s+1)\Psi$
- $S_\pm = S_x + iS_y$ $\rightarrow$ raising/lowering spin operator which preserves s and reduce/increase m_s by one unit of $\hbar$ (remember to normalize them after computations!)
- if two particles have spin s and s' then $s^{\text{tot}} = \{s + s'; s + s' - 1; ...; s - s'\}$, $m^{\text{tot}} = m_s + m'_s$
- spin 1/2: $S_i = \frac{\hbar}{2}\sigma_i$ where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
  - eigenstates in $\hat{S}_z$ basis: $|\uparrow\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $|\downarrow\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $|\uparrow\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$; $|\downarrow\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
  - 2 spin 1/2 in singlet (*antisymmetric*) config. with $s = 0$ and triple (*symmetric*) config. with $s = 1$
- $\vec{J} = \vec{S} + \vec{L}$: all possible values are $\{l + s; l + s - 1; ...; l - s\}$ $\rightarrow$ highest: *parallel* & *aligned*; lowest: *antiparallel*
- Recall: when you have $S_1 \cdot S_2 / S \cdot L$ remember that: $A_1 \cdot A_2 = \frac{(A_1 + A_2)^2 - A_1^2 - A_2^2}{2}$
- spectroscopic notation is given by ${}^{2s+1}L_j$ where $L \in \{S; P; D; F\}$ corresponding respectively to $l = \{0; 1; 2; ...\}$
- spin and spatial operators always *commute*!

5.5 Approximation methods

- Variational principle: choose set of possible wavefunctions in terms of parameter $k \to \min\langle E_k\rangle$ is *upper* bound on E_0 where $\langle E_k\rangle = \langle\psi_k|H\psi_k\rangle$
- T.I.P.T.: given $H = H^0 + \lambda H'$ to 1$^{\text{st}}$ order $E'_n = \langle\psi^0_n|H'\psi^0_n\rangle$; $\psi'_n = \sum_{n\neq m}\langle\psi^0_n|H'\psi^0_n\rangle/(E^0_n - E^0_m)\psi^0_m$
  - to 2$^{\text{nd}}$ order $E'_n = \sum_{m\neq n}|\langle\psi^0_n|H'\psi^0_n\rangle|^2/(E^0_n - E^0_m)$
  - if states are *degenerate* create matrix W where each element $W_{ij} = \langle\psi^0_i H'\psi^0_j\rangle$
    eigenvalues of W are E'_s corrections;
    eigenvectors are good linear combination of *unperturbed* states
- Adiabatic transformation (slowly change H to H'): final energy determined by corresponding eigenstate of Hamiltonian with *new* parameter
- Sudden change: energy/wavefunction stays *constant* $\Rightarrow$ for S.H.O. when $\omega \to \alpha\omega$: $\{V, T\} \to \{\alpha^2 V, T(\text{kinetic})\}$

5.6 Many particles systems

- *distinguishable* particles: $\Psi = \Pi_i\psi_i$ with $H\Psi = E\Psi$ and $E = \sum_i E_i$
- *indistinguishable*: labels are not physical so swapping makes *no* difference
  - Bosons (*symmetric* solution): integer spins and as many as you want in one state
  - Fermions (*antisymmetric* solution): half-integer spins and only 1 per state
  - Recall the symmetry is given by *spatial* +*spin* wavefunction
  - when adding n-spins, the *highest* spin state ($s = \frac{n}{2}$) is always *symmetric*

6 Special Relativity

6.1 Foundations

- In all inertial reference frames: speed of light is *constant*; laws of physics are *identical*
- for $\gamma = 1/\sqrt{1-\beta^2}$ with $\beta = \frac{v}{c}$ then a S' frame that moves with relative speed v has coordinates:
  $t' = \gamma(t - \frac{v}{c^2}x)$; $x' = \gamma(x - vt)$ $\rightarrow$ inverse transformation is the same with the + sign
  for $v \lll c \rightarrow \gamma \approx 1 + \frac{v^2}{2c^2}$; $\frac{1}{\gamma} \approx 1 - \frac{v^2}{2c^2}$
- time dilation: $\Delta t = \gamma \Delta t'$ (to derive this recall to *fix* x' and *not* x) $\rightarrow$ time is *slower* in rest frame so if you are given the time in this frame in the lab *more* time has passed
- length contraction: $L' = \gamma L$ (to derive this *fix* t and *not* t') $\rightarrow$ objects moving are shortened by a factor γ
- object moving relative to another with speed v in $x-$ direction: $u'_x = \frac{u_x + v}{1+u_x v/c^2}$; $u'_y = \frac{u_y}{\gamma(1+u_x v/c^2)}$; $u'_z = \frac{u_z}{\gamma(1+u_z v/c^2)}$
- Lorentz transformation for *boost* along x-axis (easily generalizable to any other axis):
- $p^\mu = (E/c; \vec{p})$ with $\vec{p} = \gamma m \vec{v}$ (energy-momentum vector);
  $J^\mu = (c\rho; \vec{J})$ (current density vector); $k^\mu = (\omega/c; \vec{k})$ (wave vector)
- $E = T + E_0 = \gamma mc^2$ where $E_0 = mc^2$ (*rest* energy) and $T = (\gamma - 1)mc^2$ (kinetic energy)
- The relativistic product $a \cdot b = a^0 b^0 - \sum_i^3 a_i b_i$ is *invariant* under Lorentz transformation (equal in all ref. frames)
- the invariant 4-vector displacement $(\Delta x)^2 = (x^\mu_B - x^\mu_A)^2$
  - $(\Delta x)^2 > 0$ *timelike*: there exists an inertial frame where they both appear in the same *place* ($v < c$)
  - $(\Delta x)^2 < 0$ *spacelike*: there exists an inertial frame where they both appear in the same *time* ($v > c$)
  - $(\Delta x)^2 = 0$ *lightlike*: trajectory going at speed of light ($v = c$)
- relativistic Doppler shift only depends on *relative* velocity between source and observer $v = \beta c$ such that:
  - $\frac{\lambda'}{\lambda} = \sqrt{\frac{1\pm\beta}{1\mp\beta}}$; $f' = \frac{c}{\lambda'} = c\sqrt{\frac{1\mp\beta}{1\pm\beta}} f$ $\rightarrow$ + or - in the numerator respectively tell us that the source is moving *away* and *towards* us
  - if speed of light is mentioned we are in the relativistic regime hence think of Doppler shift in these terms
- Lorentz transformations for E, B fields do *not* change their magnitude in the direction of motion of the particle
- when analyzing a system sometimes easier to take $c = 1$

6.2 Collisions

- $\boxed{E^2 = \vec{p}^2 c^2 + m^2 c^4}$ since $p^2_{4\text{d}} = m^2 c^2$ (4-vector p^μ squared) in all inertial frames
- if no ext. force: $\sum_i p^\mu_i = \sum_f p^\mu_f$ all the 4-energy momentum is conserved (recall this does not mean *invariant*!)
  - tot. momentum and tot. energy: conserved but not invariant
  - kinetic energy *neither* conserved *nor* invariant
- if two objects with *same* mass and speed collide against each other, resulting product has no speed and hence *only* rest energy
- pay careful attention when you change reference frame: if in one frame A moves with v and B is at rest; in the frame where A is at rest B moves with speed $-v$
- if particle moves with $\omega = \omega'_0$ on a circular orbit, then in its frame $\omega'_0 = \frac{2\pi}{\Delta t'}$ $\rightarrow$ hence in frame at rest $\omega_0 = \frac{\omega'_0}{\gamma}$
- just act *dumb*: just apply the rules of energy-momentum conservation and the relativistic invariants

7 Atomic Physics

7.1 Photons interactions

- chemical potential of photons is $\mu = 0$: they can be created or destroyed in any process (no conservation law!)
- *photoelectric* effect (low energy): $E_{\max} = \hbar f - \Phi$
  - Φ is work function (energy required to remove an electron from atom); $E_{\max}$ is the stopping energy
  - $f_{\text{thr}} = \frac{\Phi}{\hbar}$; $T \propto f$ (kinetic prop. to frequency); $P \propto Z^4$ (probability)
- *Compton* scattering with atomic electron (medium energy): $\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$;
  - $\Delta E = h\Delta f = \frac{hc}{\Delta\lambda} = mc^2(1 - \cos\theta) \to$ the *wider* angle the more energy *loses* electron
  - *Compton* wavelength is $\lambda = \frac{h}{mc}$: wavelength of photon whose energy is same as mass of particle
  - $P \propto Z$ (probability)
- electron-positron *pair* production (high energy-$E_r > 2m_e c^2$) $\to \vec{E}$ near nucleus induces the process there is no *reverse* reaction and the probability $P \propto Z^2$
- emission can be *spontaneous* (excited states *always* emit); *stimulated* (the more photons; the more are emitted at same frequency; polarization; phase) $\to$ amplitude $A^2 \propto (N + 1)$ where $N = \#$ photons; $\omega = \frac{(E_2 - E_1)}{\hbar}$
- absorption has amplitude $A^2 \propto N$ where $N = \#$ photons; $\omega = \frac{(E_2 - E_1)}{\hbar}$
- $N = \#$ photons$=E_{\text{tot}}/\left(\frac{hc}{\lambda}\right)$ where $E_{\text{tot}} = P\Delta t \to$ recall that $p = \frac{h}{\lambda}$ then $\lambda = \frac{h}{mv}$ (in order to go from λ to v when you don't know the frequency)
- *Lasers* keep lots of electrons in excited state through an optical pump causing *population inversion*
  - spontaneous + stimulated emission from atoms cause *cascade* of electrons which excite other atoms and cause exponential production of photons all *coherent*; *monochromatic*; high intensity
  - since excited state decays very fast: *metastable* state introduced between the two
  - Diode: medium p-n junction injected with current; Solid state: medium is *crystal*; Dye: medium is *liquid*
  - Gas: collisional (transitions from *collision* btw. atoms); molecular (transitions are *vibrational* energy levels)
  - free electrons: in ext. $\vec{E}$ field they emit *bremsstrahlung* producing *synchotron* radiation in a *sinusoidal* path $\to$ radiation produced from slowing down of electrons due to nuclear attraction
- Cherenkov radiation: results when charged particle (usually electron) travels through dielectric at speed *faster* than that at which *light* is propagating in the medium

7.2 Nuclei properties

- masses of atoms $\boxed{\sim 10^{-31}\text{kg}}$; nuclear size $\boxed{10^{-15}\text{m}}$
- Binding energy: $\text{BE} = \sum_i m_i c^2 - Mc^2$ (difference between mass of constituents and nucleus itself)
  - much *larger* than energy holding electrons together (per nucleon $\sim$ a few MeV for most elements)
  - BE per nucleon steadily increases with Z and then decreases for radioactive atoms: $Z = 26$ iron is most stable atom; $Z > 82$ all nuclei will eventually decay
  - resulting *kinetic* energy given by *difference* in binding energy between initial and final state
- for light elements # neutrons=#protons; for *heavier* elements # neutrons>#protons
- fission & fusion: *spontaneous* if mass of reactants is *larger* than mass of products
  - enormous energy to overcome (electromagnetic repulsion between protons)
  - *generates* enormous amount of energy

- energy to remove one electron is *ionization* energy → for hydrogenic atoms $E = Z^2 E_0$
  - when last orbital is: full (noble gases) or almost full (alogens) *high* ; almost empty (alkali metals) *low*
  - when they ask for electron charge distribution they mean the *valance* band → think of what its corresponding wavefunction looks like and its symmetries
  - energy scale of atomic processes is a few $\sim$ eV: use it to approximate ionization energies of *hydrogenic* atoms
- penetration depth is when $\frac{1}{2}m\dot{r}^2 = V(r) = \frac{kZq^2}{r^2}$ where → for two atoms with different Z: $V(r) = \frac{kZ_1Z_2q^2}{r^2}$
- Recall: absorption and emission *lines* are always due to spin *splitting* (nothing to do with nuclear interactions)

7.3 Interaction of charged particles

- *cross section* A defines effective collision probability: $A = P\frac{V}{N}\frac{1}{\tau}$ where P is prob. of being scattered; $\frac{V}{N}$ is concentration of targets and τ is the thickness
  - can be also thought as area of the shadow (area of sphere from distance from the target)
  - usually just think of data provided and do *dimensional* analysis
- nuclei target almost *exclusively* atomic electrons → energy loss: only through collisions in very small amount (*continuous* flow as they interact); path shape: *straight lines*; avg. path length: 10^{-5}m
- electrons target *both* nuclei and electrons → energy loss: through collisions/radiation; path shape: *scattered* at various angles; avg. path length: 10^{-3}m
- Decays: Alpha (α) → spontaneous decay of 2 neutrons + 2 protons
  - Beta decay (weak-force decay): β^- ⇒ emits electron and antineutrino; produces proton ($n \to p + e^- + \bar{\nu}_e$)
    β^+ ⇒ emits positron and neutrino; produces neutron ($p \to n + e^+ + \nu_e$)
    neutrino are responsible for *broad* energy spectrum
  - *gamma* (γ) radiation: emission of photons from excited state of nucleus which doesn't change proton/neutron composition
  - Internal conversion (IC): excited nucleus interacts with electron on lower atomic orbital causing its *emission* → produces several *x-rays*
  - Radioactive: decays randomly *independently* of how long it's been around (Poisson distribution)
    $N = N_0 e^{-\frac{t}{\tau}}$ with $t_{\frac{1}{2}} = \tau \ln 2$ and τ = mean life ⇒ prob. of seeing *zero* events is $P(0) = e^{-\frac{t}{\tau}}$
    if substance can decays in many different ways tot. half time: $1/t_{\frac{1}{2}}^{\text{tot}} = \sum_i 1/t_{\frac{1}{2}}^{i}$

8 Specialized & Miscellaneous Topics

8.1 The Standard model of particles

- Weak force: W^+ and Z bosons (very *heavy* ~ 90x m_p) mediate;
  Leptons (electron/neutrino) interact with force; $\Rightarrow$ also interact with EM force
  quark interact also via weak force and can change flavor by emitting or absorbing W-boson
  decay time $\boxed{\sim 10^{-8}\text{s}}$; signature is emission of neutrino
- EM force: photons (*massless* bosons with $s = 1$) mediate; decay time $\boxed{\sim 10^{-17}\text{s}}$; signature emission of photon
- Strong force: gluons (*massless* bosons mediate) $\rightarrow$ decay time $\boxed{\sim 10^{-23}\text{s}}$
  - *Hadrons* interact with force: bosons (composed by quark-antiquark pairs with $s = \{0, 1\}$) are called *mesons*; fermions (composed by 3 quarks with $s = \{\frac{1}{2}, \frac{3}{2}\}$) are called *baryons*
  - protons (2 quark up-1 quark down); neutron (2 quark down-1 quark up) are called *nucleons*
  - it involves *color* (corresponding charge of the force) $\rightarrow$ this was able to explain existence of 3 up/down quarks together without violating the Pauli principle
  - blue, green, red: together they make the *white* which means the charge is 0 and particle is color *neutral*
  - given *confinement* property of strong force, free quarks cannot be seen in nature
- matter organized in 3 generations: each of them is *heavier* and *less* stable $\Rightarrow$ 2$^{\text{nd}}$, 3$^{\text{rd}}$ gen. decay to 1$^{\text{st}}$
  - 1$^{\text{st}}$ gen.: up quark ($+\frac{2}{3}$); down quark ($-\frac{1}{3}$); electron; electron neutrino
  - 1$^{\text{st}}$ gen.: charm quark; strange quark; muon (~ 200x electron); muon neutrino
  - 3$^{\text{rd}}$ gen.: top quark; bottom quark; tau (~ 20x muon); tau neutrino
- every particle has an *antiparticle* with equal mass and opposite charge $\rightarrow$ photons are their own antiparticles with $s = -1$; Z are their own antiparticles and W^+ has antiparticle W^-
- to determine which force is responsible for decay one must look at combination of life-time and decay products
- in particle physics anything that can happen *will* happen unless it's forbidden by a symmetry/ conservation law:
  - baryon and lepton number conserved (recall antiparticles have -1; particles $+1$)
  - CPT symmetry: charge conjugation (C) $\Rightarrow$ switch particles with antiparticles and change sign of all charges
    time reversal (T) $\Rightarrow t \rightarrow -t$
    parity transformation (P) $\Rightarrow$ reverses orientation in space
  - supersymmetry: idea that particles have super-partners with exactly same charge and spin different by $\frac{1}{2}$
  - weak interaction is said to be *maximally* parity-violating
- Higgs boson (125GeV): responsible for giving mass to all elementary particles through mechanism of spontaneous symmetry breaking (SSB) $\rightarrow$ when system moves to a vacuum solution that exhibits the same symmetry which is broken for perturbations around vacuum and preserved for the entire lagrangian
- Recall: a neutron has non-zero *magnetic-dipole* moment but no *electric-dipole* moment $\rightarrow$ if it had it would corresponds to a parity violation
- Recall: a *freely* propagating neutrino is superposition of muon and tau neutrino

8.2 Crystal Structures

- infinite repetitions of identical structural units (*unit cells*)
- Simple cubic: atoms at every *vertex* ($d = a$); Body-centered cubic (BCC): also atom at the *center* ($d = a\frac{\sqrt{3}}{2}$); Face-centered cubic (FCC): atoms at center of each *face* ($d = a\frac{\sqrt{2}}{2}$)
- the smallest pattern is called *primitive* unit cell (not necessarily equal to unit cell)

- BCC is *octahedron* with *half* volume of unit cell
- FCC is *parallelepiped* with *quarter* volume of unit cell

- *Reciprocal* (dual) lattice is the Fourier transform of the original lattice (in *momentum* space)
  - Simple cubic is its own reciprocal lattice with length $d_p = 2\pi/a$
  - BCC and FCC are the dual lattices of each other
  - the dual of an *hexagonal* lattice is another hexagonal lattice *rotated* by 30°

8.3 Astrophysics

- scale factor $a(t)$ measures *expansion* of universe → this causes *redshift* of photons which is used as measure of *time*: $a = 1/1 + z$; $\frac{\lambda_0}{\lambda_t} = \frac{a_0}{a_t} \rightarrow \boxed{z(t) = \frac{\lambda_0}{\lambda_t} - 1}$
- Hubble's law (HL): $v = H_0 d$
  - due to expansion of space, distant objects seem to *recede* from us (think of *inflating* balloon)
  - given Hubble constant and distance use HL to find receding speed v and compute typical relativistic effect
- if universe expands by factor n; it cools down in temperature by factor n
- Neutron stars are giant spheres of neutrons (fermions): *cannot* collapse to be in same position by Pauli exclusion principle

8.4 Error Analysis

- The sample variance is $\sigma_s^2 = \frac{1}{N-1}\sum_i (x_i - \bar{x})^2$ → if *sample*: $\frac{1}{N-1}$; if *whole* population: $\frac{1}{N}$ (std. dev. is just σ_s)
- error propagation: $f = aA$: $\sigma_f = a\sigma_A$; $f = A \pm B$: $\sigma_f = \sqrt{\sigma_A^2 + \sigma_B^2}$; $f = AB \vee f = \frac{A}{B}$: $\frac{\sigma_f}{f} = \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2}$
- *independent* errors add in *quadrature* $\sigma_{\text{tot}} = \sqrt{\sigma_1^2 + \sigma_2^2}$
- inverse variance weighting: $x_{\text{avg}} = \sum_i w_i x_i / \sum_i w_i$ where $w_i = 1/\sigma_i$ and $\sigma_{\text{avg}} = 1/\sqrt{\sum_i w_i}$
- Poisson distribution: $P(n) = \frac{\lambda^n e^{-\lambda}}{n!}$
  - λ= exp. avg. number of counts in given time t and $P(n)$ is probability to observe n counts in such time t → $\sigma \approx \sqrt{N}$ if $N \ggg 1$;
  - $\sigma_{\text{avg}} = \sqrt{\mu}$ where μ is *mean* value (on tot. count, not on count rate) ⇒ after N measurements $\sigma_N = \frac{\sigma_{\text{avg}}}{\sqrt{N}}$
  - if measurements are *purely* random always use Poisson
  - the time between two Poisson events follows exponential: $P(t) = \lambda e^{-\lambda t}$
  - 90% confidence limit means we want to find rate that gives 0 with probability 0.1 (only 10% of the time): $P(0) = e^{-\lambda t} = 0.1$
- Recall: *accuracy* means how *far* from true value ; *precision* means how *reproducible* the result is (variance)
- errors can be *systematic* (cannot be reduced); *statistical* (can be reduced by repeating experiments)

9 General Tips for the Exam

- Always be very careful with signs!$\rightarrow$ think about computing *unsigned* quantity and put sign just at the end.
- if a problem doesn't give you a quantity that you thought you would need: *think!* it's probably not useful (usually it means that some other quantity is conserved)
- Remember to *always* exhaust all limiting cases and dimensional analysis before doing any algebra
  - if you have choices with different dimensions always check them first, it may be enough!
  - look at *orders* of magnitude to build some intuition
  - use the units in the solutions to figure out if limiting cases can help remove possibilities
- answers that have numerical factors/ random numbers: *slow down* and work it out carefully!
- if the answer is *wrong*; it's just *wrong*! $\rightarrow$ there are never *typos* in the exam!
- Recall generally *never* is too strong of a word to be favored by ETS: probably that choice is wrong!
- Always *guess*: there is no penalty for wrong answers!

9.1 Useful Math

- log plots: check if one or both axes are in log. scale
  - check plots to see if: axis starts at 1 and *not* zero; squares that separate points are not *equally* spaced
  - straight line: on log-log $y = ax^b$; on log-plot $y = c \cdot 10^{bx}$ (here we assumed x-linear)
- Always read axis to verify if they carry *dimensions*
- $e^x \approx 1 + x$; $(1+x)^n \approx 1 + nx$; $\sum_{n=1}^{N} n = \frac{n(n+1)}{2}$; $\sum_{n=1} x^n = \frac{1}{1-x}$ for $|x| < 1$
- Fourier transforms: $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$ $\rightarrow$ when asked about its coefficients think of symmetry of function: if *even* no sin terms (odd!); if *odd* no cos terms (even!)
- if an event occurs with probability P and I want to make sure it does *not* happen N times: $\bar{P}_N = (1-P)^N$
- Don't forget Stoke's theorem: $\int_S \nabla \times \mathscr{U} \cdot d\vec{a} = \int_C \mathscr{U} \cdot d\vec{l}$

9.2 Numbers to memorize

- 13.6eV (E_0 of hydrogen)
- 511keV (electron mass): whenever you see this number think of electrons
- 1.22 (Rayleigh criterion coefficent)
- $2.9 \cdot 10^{-3}\text{m} \cdot \text{K}$ (proportionality factor between λ and T)
- 2.7K (CMB temperature)