The Schwinger Effect

Overview and extended derivation

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This paper represents my own work in accordance with University regulations.

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1 Introduction

The Schwinger effect is a non-perturbative phenomenon by which electron-positron pairs can be produced in vacuum by a static electrical field. Sauter was the first to find out in 1931 that the Dirac vacuum became unstable against the pair creation of electrons and positrons, [1] but it is only in 1950 that this production rate Γ per unit time and volume was rigorously computed by Schwinger, in the context of quantum electrodynamics. [2]

$$\Gamma = \frac{e^2 E^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{n\pi m^2}{eE}} \tag{1}$$

The non perturbative nature of this phenomenon is shown in the fact that the electromagnetic field is time-independent and that the corresponding transition amplitudes are proportional to $\exp(-\frac{\pi m^2}{eE})$[1] which implies that the effect can never shows up at any fixed order in perturbative QED. [3]

As shown in eq.1, the external field must be very intense in order to lead to a significant probability for particle production. The rate of pair production becomes of order 1 only for $E \sim E_c = m^2 e \approx 10^{18}$ V/m which is an enormous field that still surpass by several orders of magnitude the largest fields achievable experimentally. [3]

Although not empirically confirmed, the Schwinger effect has been proven theoretically through several methods in quantum field theory: from the Wigner formalism to Bogoliubov transformations. [4, 5]

In this paper, I will at first use a quantum mechanical approach through the WKB-approximation to provide an intuitive explanation of the nature of this effect as a tunnelling process. [6] Then, I will rigorously derive the effect using the original's Schwinger method through the in-out formalism with the effective action.

[1]Note, for simplicity in the notation, throughout this paper I always assume $\hbar = c = 1$, unless stated otherwise.

2 The WKB Approximation

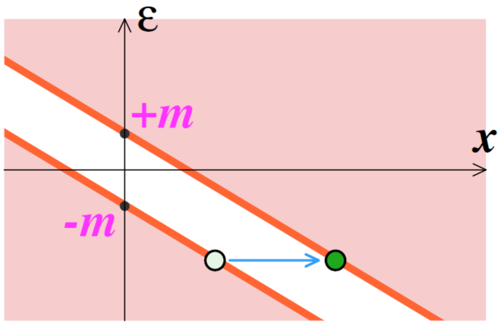


Figure 1: Schematic picture of the tunneling process involved in the Schwinger mechanism. The white band is the gap between the anti-electron Dirac sea and the positive energy electron continuum, tilted by the potential $V(x) = -Ex$ in the presence of an external electric field E. [3]

To understand how the WKB-approximation method works in this context, it is crucial to at first introduce the Dirac sea picture for the energy solutions that arise from the Dirac equation. In fact, this equation gives both positive and negative energy solutions, a property that was at first seen as a problem since one has to explain why a positive energy wouldn't decay into a negative energy solution by a continuous emission of photons. To solve this puzzle, we can indeed think of these energy solutions as part of the "Dirac" sea, according to which the vacuum of the theory consists of a completely filled, negative energy band as well as an empty, positive energy band, separated by an energy gap of $2m$. A hole in the negative energy band is then simply interpreted as the electron's antiparticle, the positron.

According to this view, the behavior of the electron-positron pair transition probabilities can be understood as a tunneling phenomenon by which a particle from the Dirac sea is pulled into the positive energy states.

In fact, as shown in fig.1, a level crossing occurs between the positive and negative energy bands once they are tilted by the potential $V(x) = -|\vec{E}|x$ in the presence of a constant external electric field $\vec{E}$. This level crossing then allows a negative energy electron to tunnel through the energy gap to the positive energy band, leaving a hole behind. [7]

One can thus get an estimate for the pair production rate using the WKB approximation tunnelling formalism. To do this, first of all we need to define the potential across which the particles are moving. If we take the center of nucleation at $x = 0$ then the positron e^+ and electron e^- move respectively in the potential: [6]

$$V_{e^\pm}(x) = m \pm e|\vec{E}|x \quad \text{if } x > 0 \tag{2}$$

where as expected electron and positron travel in opposite direction in x.

In the WKB approximation, the transmission probability coefficient for a particle tunnelling is proportional to the exponential of $\frac{2I}{\hbar}$, where I represents the integral of the particle's momentum across the barrier $p(x) = \sqrt{2m(E - V(x))}$. [8] This description is only valid for classical trajectories which in our case, are not available. Thus we need to use the semiclassical interpretation of the tunneling effect which basically consists in turning the potential upside down or, in other words, performing a transformation to imaginary momentum in the classically forbidden region, such that:

$$p(x) \rightarrow ip(x) = \sqrt{2m(V(x) - E)}$$

The appropriate solution for the tunneling rate can then be written as:

$$T = \exp\Big\{ -2 \int_0^{x_0} \sqrt{2m(V(x) - E)} dx \Big\}$$

where, in the case under consideration, for a positron and a electron with no initial energy, their (imaginary) momentum is given by the relativistic dispersion relation such that: [6]

$$I_{e^\pm} = \int_o^{x_0} p_{e^\pm} = \int_0^{x_0} \sqrt{V_{e^\pm}(x)[2m - V_{e^\pm}(x)]} dx$$

x_0 represents the upper limit of the barrier and it can computed by making a simple consideration in relation to the Heisenberg principle. The emission of the pair from vacuum has an upfront cost of $2m$, which can only be repaid once the pair are far enough apart, $\Delta x > 2x_0$ where x_0 is hence equal to $\frac{m}{e|\vec{E}|}$. [6] Finally since the transition probability must describe the motion of both the positron and the electron, we need to sum over the contribution from the respective integrals of the momenta $I_\pm$. Thus:

$$T = \exp\Big\{ -2\Big[I_{e^+} + I_{e^-}\Big]\Big\} = \exp\Big\{ -4 \int_0^{x_0} \sqrt{(m - e|\vec{E}|x)(m + e|\vec{E}|x)} dx \Big\}$$
$$= \exp\Big\{ -4m \int_0^{x_0} \sqrt{1 - \frac{x^2}{x_0^2}} dx \Big\} = \exp\Big\{ -\frac{m^2}{e|\vec{E}|} \Big\}$$

Comparing this result to the production rate Γ in eq.1, one can note that T corresponds exactly to the exponential decay factor of the first and dominant term in the n summation of Γ, thus providing a fairly good approximation for the total production rate.

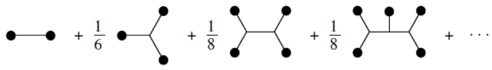
3 S-Matrix and the QED effective action

In Quantum field theory, all the information about an autonomous quantum system is contained in its unitary evolution operator $\hat{U}$ or S-matrix defined to map an initial state to its final state according to the Hamiltonian H of the system:

$$\hat{U}(t_f, t_i) = T \exp\Big[-i \int_{t_i}^{t_f} \hat{H}(t') dt' \Big]$$
$$\hat{S} = \lim_{t_f, t_i \rightarrow \pm\infty} \hat{U}(t_f, t_i)$$

The scattering amplitudes $\langle \mathbf{b}|\hat{S}|\mathbf{a}\rangle$ determine the probabilities for the system to evolve from its initial state $|\mathbf{a}\rangle$ to its final state $|\mathbf{b}\rangle$. Thus, the probability for the vacuum to be stable also named the vacuum persistence probability amplitude can be thought as the square of $S_0 = \langle \mathbf{0}_{out}|\mathbf{0}_{in}\rangle = \langle \mathbf{0}_{in}|\hat{S}|\mathbf{0}_{in}\rangle$. A standard result in field theory is that this vacuum transition amplitude can be rewritten as the exponential of the sum of all the connected vacuum diagrams [9]:

$$\langle \mathbf{0}_{out}|\mathbf{0}_{in}\rangle = e^{i\mathcal{V}}; \quad \nearrow \quad i\,\mathcal{V} \equiv \tag{3}$$



After squaring this amplitude, we get $P_0 = \exp[-2\Im(\mathcal{V})]$. In the presence of an external source that remains constant over a long period of time and that is sufficiently spatially homogeneous (in our case a constant electric field E), the imaginary part $2\Im(\mathcal{V})$ can be written in the form of an integral over space-time $2\Im(\mathcal{V}) = \int d^4x(...)$, whose integrand can be interpreted as the particle production rate per volume per unit time. [3] It becomes thus clear that, in order to evaluate the electron-positron pair production rate, one must find S_0.

A conventional method for computing the S-matrix of the QED theory is to exploit the functional integral formalism and rewrite $\hat{S}$ as a path integral such that, given a constant electromagnetic field A_μ [9]:

$$S_0 = Z_0[A_\mu] = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi e^{iS[\psi,\overline{\psi}]} \tag{4}$$

where $S[\psi,\overline{\psi}] = \int dx^4 \mathcal{L}_{\text{QED}}$ represents the QED action. To evaluate the path integral in eq.4 it is useful to introduce the effective action $\Upsilon[A_\mu]$[2] so that $S_0 = e^{i\Upsilon}$.[3] According to eq.4, such effective action can be defined as [10]:

$$\int \mathcal{D}A \exp(i\Upsilon[A_\mu]) = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left[i\int dx^4\mathcal{L}_{\text{QED}}\right] \tag{5}$$

Now, given that $\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, one can note that the only term in the Lagrangian that depends on the fermionic field ψ is that involving the Dirac operator $\not{D}$, whose path integral is known [9] and equals to:

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left[i\int dx^4(\overline{\psi}(i\not{D} - m)\psi)\right] = \mathcal{N}\text{det}(i\not{D} - m) \tag{6}$$

where $\mathcal{N}$ is some infinite normalization constant which later will be dropped out. Plugging in the result from eq.6 into the right hand side of eq.4 one can write:

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left[i\int dx^4\overline{\psi}\left(i\not{D}_\mu - m\right)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\right] = e^{-\frac{i}{4}\int dx^4F^{\mu\nu}F_{\mu\nu}} + \mathcal{N}\text{det}(i\not{D} - m) \tag{7}$$

Hence,

$$\begin{aligned}
\int \mathcal{D}A \exp(i\Upsilon[A_\mu]) &= \int \mathcal{D}A e^{-\frac{i}{4}\int dx^4F^{\mu\nu}F_{\mu\nu}} + \mathcal{N}\text{det}(i\not{D} - m) \\
\Rightarrow i\Upsilon[A] + \frac{i}{4}\int dx^4F^{\mu\nu}F_{\mu\nu} &= \ln[\det(i\not{D} - m)] + \ln\mathcal{N} \\
&= \text{Tr}[\text{tr}(\ln(i\not{D} - m))] + \ln\mathcal{N}
\end{aligned}$$

Where Tr specifies a Dirac trace and then I used the fundamental property for all $n \times n$ matrices B over $\mathbb{C}$ which states that $\ln(\det B) = \text{tr}(\ln B)$. [10] A trace is a sum over all eigenvalues, in this case of $(i\not{D} - m)$, and its value is basis independent. This means that one has the freedom to evaluate it in the space of preference. Here, I will compute the sum on the position eigenstates, using the general formula for a matrix B:

$$\text{tr}B = \int dx^4\langle x|B|x\rangle \tag{8}$$

[2]Generally the effective action is associated to the symbol Γ, in this paper to not create confusion with the particle production rate Γ, I will always refer to it as Υ.

[3]Note that we are allowed to assume this form given the unitarity of the S-matrix $SS^\dagger = S^\dagger S = 1$, which simply expresses the conservation of probability.

As a result $i\Upsilon[A]$ becomes:

$$\begin{aligned} i\Upsilon[A] = i\int dx^4 \mathcal{L}_{\text{eff}} = i\int dx^4\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\Big) + \text{Tr}[\langle x|\ln(i\not{D}-m)x|\rangle] \\ \Rightarrow \mathcal{L}_{\text{eff}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\text{Tr}[\langle x|\ln(i\not{D}-m)x|\rangle] \end{aligned} \tag{9}$$

In eq.9, I defined the effective Lagrangian density $\mathcal{L}_{\text{eff}}$ that arises from $\Upsilon[A]$ and I dropped the the infinite normalization constant $\mathcal{N}$, exploiting the fact that $\mathcal{L}_{\text{eff}}$ can be always shifted by a constant to remove infinities when it's renormalized. [10]

Since the trace of an operator is invariant under transposition, given that the charge conjugation matrix C satisfies $C\Upsilon_\mu C^{-1} = -\Upsilon_\mu^T$, one can notice that $\text{Tr}[\langle x|\ln(i\not{D}-m)x|\rangle] = \text{Tr}[\langle x|\ln(-i\not{D}-m)x|\rangle]$, [10] hence to calculate its value it's sufficient to average the two to get:

$$\text{Tr}[\langle x|\ln(i\not{D}-m)x|\rangle] = \frac{1}{2}\text{Tr}[\langle x|\ln(-\not{D}^2-m^2)x|\rangle] \tag{10}$$

Plugging this result into $\mathcal{L}_{\text{eff}}$ and taking its derivative with respect to m^2 one gets:

$$\begin{aligned} \frac{d}{dm^2}\mathcal{L}_{\text{eff}} = 0 - i\frac{d}{dm^2}\Big(\frac{1}{2}\text{Tr}[\langle x|\ln(-\not{D}^2-m^2)x|\rangle]\Big) \\ = \frac{i}{2}\text{Tr}\Big[\langle x|\frac{1}{(-\not{D}^2-m^2)}|x\rangle\Big] \end{aligned} \tag{11}$$

By restoring the Feynman prescription $i\epsilon$ in $\frac{1}{-\not{D}^2-m^2}$, one can exploit a useful identity defined in [2]:

$$\begin{aligned} \frac{1}{A+i\epsilon} = \int_0^\infty ds(e^{is(A+i\epsilon)}) \\ \Rightarrow \frac{1}{-\not{D}^2-m^2+i\epsilon} = \int_0^\infty ds(e^{is(-\not{D}^2-m^2+i\epsilon)}) \end{aligned} \tag{12}$$

where s is the so-called Schwinger's proper time. This identity allows us to rewrite $\frac{d}{dm^2}\mathcal{L}_{\text{eff}}$ in the integral form:

$$\frac{d}{dm^2}\mathcal{L}_{\text{eff}} = \frac{1}{2}\int_0^\infty dse^{-ism^2-s\epsilon} \times \text{Tr}\Big[\langle x|e^{-i\not{D}^2 s}|x\rangle\Big] \tag{13}$$

In integrating over m^2 one notes that:

$$\int\Big(\int_0^\infty dse^{-ism^2}\Big)dm^2 = -i\int_0^\infty \frac{ds}{s}e^{-ism^2} \tag{14}$$

Consequently, $\mathcal{L}_{\text{eff}}$ becomes:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2} \times \text{Tr}\Big[\langle x|e^{-i\not{D}^2 s}|x\rangle\Big] + \text{const} \tag{15}$$

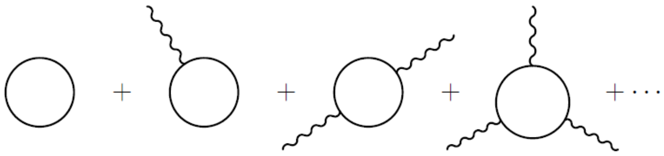
where, in eq.15 I also dropped the term $e^{-s\epsilon}$ since it's essentially negligible for $\epsilon \lll 1$.

Before moving on with the computation, it is useful to reflect on the physical meaning of $\mathcal{L}_{\text{eff}}$ as presented in eq.15. In QED, the Dirac propagator $G(x,y)$ in the presence of an external field A_μ can be written as [11]:

$$\begin{aligned} G_A(x,y) = \langle y|\hat{G}|x\rangle \\ \text{where } \hat{G}_A = \frac{i}{(\not{D}-m+i\epsilon)} = (\not{D}+m)\frac{i}{\not{D}^2-m^2+i\epsilon} \end{aligned} \tag{16}$$

By rewriting $G_A(x,y)$ using the proper time integral formalism of eq.12, one can see that up to a factor of $(\not{D}+m)/s$, the proper time integral in eq.15 looks just like $G_A(x,y)$ for $x=y$. [10] Hence, we can interpret the effective action as the sum of all closed fermion one-loop diagrams with increasing number of external photons: [10]

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}F^2_{\mu\nu} + \cdots \qquad (17)$$



4 The Euler-Heisenberg Lagrangian

In this section, I will derive the Euler-Heisenberg Lagrangian $\mathcal{L}_{EH}$ [12] for a constant background electromagnetic field $F_{\mu\nu}$ which directly arises from our formulation of $\mathcal{L}_{\text{eff}}$ as in eq.15. This will serve as an intermediate step in the evaluation of the effective action for the case of a constant background electric field E, which can be simply viewed as a special case of $\mathcal{L}_{EH}$.

First of all to compute $\mathcal{L}_{\text{eff}}$, I will substitute $\not{D}^2$ with $D^2_\mu + \frac{e}{2}F_{\mu\nu}\sigma^\mu$, which can then be interpreted as the Hamiltonian operator $\hat{H}$.

The identity $\not{D}^2 = D^2_\mu + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu}$ comes from multiplying the Dirac equation for a fermionic field ψ by $(i\not{D}+m)$ on both sides to get:

$$\begin{aligned} (i\not{D}+m)(i\not{D}-m)\psi = (-\not{D}^2-m^2)\psi = 0 \\ = (i\not{\partial}-e\not{A}+m)(i\not{\partial}-e\not{A}-m)\psi = [(i\partial_\mu - eA_\mu)(i\partial_\nu - eA_\nu)\Upsilon^\mu\Upsilon^\nu - m^2]\psi \\ = \frac{1}{4}\Big(\{i\partial_\mu - eA_\mu, i\partial_\nu - eA_\nu\}\{\Upsilon^\mu,\Upsilon^\nu\} + [i\partial_\mu - eA_\mu, i\partial_\nu - eA_\nu][\Upsilon^\mu,\Upsilon^\nu] - 4m^2\Big)\psi \\ = \Big((i\partial_\mu - eA_\mu)^2 - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2\Big)\psi = \Big(-D^2 - \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} - m^2\Big)\psi \Rightarrow \not{D}^2 = D^2_\mu + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \end{aligned} \qquad (18)$$

where to go from the 3rd to 4th line of eq.18 I used the following identities:

$$\begin{aligned} \{\Upsilon^\mu,\Upsilon^\nu\} = 2g^{\mu\nu}\mathbf{1} \Rightarrow \{i\partial_\mu - eA_\mu, i\partial_\nu - eA_\nu\}\{\Upsilon^\mu,\Upsilon^\nu\} = 4(i\partial_\mu - eA_\mu)^2 \\ [\Upsilon^\mu,\Upsilon^\nu] = -2i\sigma' \quad [i\partial_\mu - eA_\mu, i\partial_\nu - eA_\nu] = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = -ieF^{\mu\nu} \end{aligned} \qquad (19)$$

We can hence rewrite $\mathcal{L}_{\text{eff}}$ in the following form:

$$\begin{aligned} \mathcal{L}_{\text{eff}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2} \times \text{Tr}\Big[\langle x|e^{-i\hat{H}s}|x\rangle\Big] + \text{const} \\ \text{where for } i\partial_\mu \to \hat{p}^\mu \Rightarrow \hat{H} = \hat{D}^2_\mu + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} = (\hat{p}^\mu - eA_\mu(\hat{x}))^2 + \frac{e}{2}F_{\mu\nu}(\hat{x})\sigma^{\mu\nu} \end{aligned} \qquad (20)$$

The first step in the evaluation of $\mathcal{L}_{\text{eff}}$ is therefore computing $\langle y|e^{-i\hat{H}s}|x\rangle$, once this is known I will simply set $y=x$ and integrate over s. By definition $|x,s\rangle = e^{i\hat{H}s}|x\rangle$, hence one can write:

$$\partial_s\langle y,0|x,s\rangle = \partial_s\langle y|e^{-i\hat{H}s}|x\rangle = \langle y|e^{-i\hat{H}s}\hat{H}|x\rangle \qquad (21)$$

Therefore, it becomes clear from eq.21 that if we can write $\hat{H}$ in terms of the positions operators $\hat{x}(s)$ and $\hat{x}(0)$, one could turn eq.21 into an ordinary differential equation whose solution is $\langle y|e^{-i\hat{H}s}|x\rangle$.

To achieve this goal, I will present Schwinger's original method and thus introduce the operator $\hat{\Pi} = \hat{p}^\mu - eA_\mu(\hat{x})$. Its commutation relation with $\hat{x}$, assuming a constant $F_{\mu\nu}$, is given by $[\hat{x}^\mu, \hat{\Pi}^\nu] = -ig^{\nu\mu}$, [2] and it simply arises from the generalization in 4d of the well know position and momentum operator commutator: $[\hat{x}^\mu, \hat{p}^\nu] = -i \overset{4\text{d}}{\longmapsto} -ig^{\nu\mu}$. In contrast with $\hat{p}$, the commutation relation of $\hat{\Pi}^\mu$ with itself is non-zero and comes straightforwardly from its definition: $[\hat{\Pi}^\mu, \hat{\Pi}^\nu] = -ieF^{\mu\nu}$. [2]

In terms of $\hat{\Pi}^\mu$, the Hamiltonian operator becomes:

$$\hat{H} = -\hat{\Pi}_\mu\hat{\Pi}^\nu + \frac{e}{2}F_{\mu\nu}(\hat{x})\sigma^{\mu\nu} = -\mathbf{\Pi}^2 + \frac{e}{2}\mathrm{tr}(\mathbf{F}\sigma) \tag{22}$$

The Heisenberg equations generated by the Hamiltonian $\hat{H}$ for the evolution of $\hat{\Pi}_\mu(s)$ and $\hat{x}_\mu(s)$ are respectively:

$$\frac{d\mathbf{\Pi}}{ds} = i[\hat{H}, \mathbf{\Pi}] = 2e\mathbf{F}\cdot\mathbf{\Pi}, \quad \frac{d\mathbf{x}}{ds} = i[\hat{H}, x] = 2\mathbf{\Pi} \tag{23}$$

where I used the commutation relations for $\hat{\Pi}^\mu$ defined earlier and the fact that $\mathbf{F}$ is constant and hence commutes with all operators. [10] The solutions to the equations 23 are:

$$\begin{aligned} \mathbf{\Pi}(s) &= e^{2es\mathbf{F}}\mathbf{\Pi}(0) \\ \mathbf{x}(s) = \mathbf{x}(0) + \frac{e^{2es\mathbf{F}}-1}{e\mathbf{F}}\mathbf{\Pi}(\mathbf{0}) &= \mathbf{x}(0) + 2se^{es\mathbf{F}}\frac{\sinh es\mathbf{F}}{es\mathbf{F}}\mathbf{\Pi}(\mathbf{0}) \\ \Rightarrow \mathbf{\Pi}(s) = e^{es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}\cdot[\mathbf{x}(s)-\mathbf{x}(0)]; \quad \mathbf{\Pi}(0) &= e^{-es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}\cdot[\mathbf{x}(s)-\mathbf{x}(0)] \end{aligned} \tag{24}$$

The Hamiltonian then takes the form:

$$\begin{aligned} \hat{H} &= -[\mathbf{x}(s)-\mathbf{x}(0)]\frac{e^{2es\mathbf{F}}e^2\mathbf{F}^2}{4\sinh^2(es\mathbf{F})}[\mathbf{x}(s)-\mathbf{x}(0)] + \frac{e}{2}\mathrm{tr}(\mathbf{F}\sigma) \\ &= \mathbf{x}(s)\mathbf{K}\mathbf{x}(s) - 2\mathbf{x}(s)\mathbf{K}\mathbf{x}(0) + \mathbf{x}(0)\mathbf{K}\mathbf{x}(0) + K_{\mu\nu}[\hat{x}^\mu(s), \hat{x}(0)] + \frac{e}{2}\mathrm{tr}(\mathbf{F}\sigma) \end{aligned} \tag{25}$$

where to keep simplicity in our notation, we define $\mathbf{K} = \frac{e^2\mathbf{F}^2}{4\sinh^2(es\mathbf{F})}$ [2]. In the 2^{nd} line of eq.25, I then moved all the $\mathbf{x}(s)$ on the left and $\mathbf{x}(0)$ on the right picking out a factor $K_{\mu\nu}[\hat{x}^\mu(s), \hat{x}(0)]$ which can be evaluated through the commutation relations of $\hat{\Pi}$ and $\hat{x}$:

$$\begin{aligned} K_{\mu\nu}[\hat{x}^\mu(s), \hat{x}(0)] &= \mathrm{tr}\Big\{\mathbf{K}\Big[\mathbf{x}(0) + \frac{e^{2es\mathbf{F}}-1}{e\mathbf{F}}\mathbf{\Pi}(\mathbf{0}), \mathbf{x}[0]\Big]\Big\} \\ = \mathrm{tr}\Big\{\mathbf{K}\Big([\mathbf{x}[0], \mathbf{x}[0]] + \frac{e^{2es\mathbf{F}}-1}{e\mathbf{F}}[\mathbf{\Pi}[0], \mathbf{x}[0]]\Big)\Big\} &= \mathrm{tr}\Big\{\mathbf{K}\Big(i\frac{e^{2es\mathbf{F}}-1}{e\mathbf{F}}\Big)\Big\} \\ = \mathrm{tr}\Big\{i\frac{e^2\mathbf{F}^2}{4\sinh^2(es\mathbf{F})}\cdot\frac{e^{es\mathbf{F}}2\sinh(es\mathbf{F})}{e\mathbf{F}}\Big\} = i\mathrm{tr}\Big\{\frac{e\mathbf{F}e^{es\mathbf{F}}}{2\sinh(es\mathbf{F})}\Big\} &= \frac{i}{2}\mathrm{tr}\Big\{\frac{e\mathbf{F}2e^{2es\mathbf{F}}}{e^{2es\mathbf{F}}-1}\Big\} \\ = \frac{i}{2}\mathrm{tr}\Big\{\frac{e\mathbf{F}[(e^{2es\mathbf{F}}+1)+(e^{2es\mathbf{F}}-1)]}{e^{2es\mathbf{F}}-1}\Big\} &= \frac{i}{2}\mathrm{tr}\Big\{e\mathbf{F}\coth(es\mathbf{F}) + e\mathbf{F}\Big\} \end{aligned} \tag{26}$$

Since $\mathrm{tr}(\mathbf{F}) = 0$, $\hat{H}$ hence becomes:

$$\hat{H} = \mathbf{x}(s)\mathbf{K}\mathbf{x}(s) - 2\mathbf{x}(s)\mathbf{K}\mathbf{x}(0) + \mathbf{x}(0)\mathbf{K}\mathbf{x}(0) - \frac{i}{2}\mathrm{tr}\{e\mathbf{F}\coth(es\mathbf{F})\} + \frac{e}{2}\mathrm{tr}\{\mathbf{F}\sigma\} \tag{27}$$

which finally allows us to rewrite eq.21 in terms of the position operators $\hat{x}(s)$ and $\hat{x}(0)$ such that:

$$\partial_s\langle y,0|x,s\rangle = -\Big\{(\mathbf{y}-\mathbf{x})\frac{e^{2es\mathbf{F}}e^2\mathbf{F}^2}{4\sinh^2(es\mathbf{F})}(\mathbf{y}-\mathbf{x}) + \frac{i}{2}\mathrm{tr}\{e\mathbf{F}\coth(es\mathbf{F})\} + \frac{e}{2}\mathrm{tr}\{\sigma\mathbf{F}\}\Big\}\langle y,0|x,s\rangle \tag{28}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the initial and final position vectors.[4] The solution to this differential equation is rather intuitive as it resembles that of an exponential with $-i\int \hat{H}ds$ at the exponent. Explicitly the general solution takes the form: [10]

$$\langle y,0|x,s\rangle = C(x,y)\exp\left\{i(\mathbf{y}-\mathbf{x})\frac{e\mathbf{F}}{4}\coth\left(es\mathbf{F}\right)(\mathbf{y}-\mathbf{x}) - \frac{1}{2}\mathrm{tr}\ln\left[\frac{\sinh(es\mathbf{F})}{e\mathbf{F}}\right] + \frac{ies}{2}\mathrm{tr}\{\sigma\mathbf{F}\}\right\} \tag{29}$$

which holds for any function $C(x,y)$. That said, we actually don't have this much freedom, $C(x,y)$ cannot be any function since its form is fixed by two more conditions on the transformations of $\langle y,0|x,s\rangle$ from the $\hat{\Pi}$ operator, which hence allow us to uniquely determine $C(x,y)$, up to a normalization factor. The two conditions are stated below as presented by Schwinger in [2], and were derived trivially from the definition of the Heisenberg-picture operators $\hat{\Pi}$, $\hat{x}$:

$$\begin{aligned}(-i\frac{d}{d\mathbf{y}} - e\mathbf{A})\langle y,0|x,s\rangle &= \langle y,0|e^{-i\hat{H}s}\Pi(s)|x,s\rangle = e^{es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}\cdot(\mathbf{y}-\mathbf{x})\langle y,0|x,s\rangle \\ (i\frac{d}{d\mathbf{x}} - e\mathbf{A})\langle y,0|x,s\rangle &= \langle y,0|e^{-i\hat{H}s}\Pi(0)|x,s\rangle = e^{-es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}\cdot(\mathbf{y}-\mathbf{x})\langle y,0|x,s\rangle\end{aligned} \tag{30}$$

Combining the conditions in eq.30 with the solution of $\langle y,0|x,s\rangle$ in eq.29, we obtain two differential equations for $C(x,y)$, specifically:

$$[-i\frac{d}{d\mathbf{y}} - e\mathbf{A} - \frac{1}{2}e\mathbf{F}(\mathbf{y}-\mathbf{x})]C(x,y) = 0; \quad [i\frac{d}{d\mathbf{x}} - e\mathbf{A} - \frac{1}{2}e\mathbf{F}(\mathbf{y}-\mathbf{x})]C(x,y) = 0 \tag{31}$$

The solution to these equations $C(x,y)$ has the form:

$$C(x,y) = C\exp\left[ie\int_x^y d\mathbf{x}'(\mathbf{A}(\mathbf{x}') + \frac{1}{2}\mathbf{F}(\mathbf{x}'-\mathbf{y}))\right] \tag{32}$$

where the integral is independent of the integration path given that $\mathbf{A}(\mathbf{x}') + \frac{1}{2}\mathbf{F}(\mathbf{x}'-\mathbf{y})$ has vanishing curl. [2] C is a normalization constant and its value is determined by matching the Feynman's propagator for a free field written according to the Schwinger's proper-time formalism, with the propagator arising from our solution for $\langle y,0|x,s\rangle$ when $\mathbf{A}\to 0$. Hence, the Feynman's propagator for a free field $G_F(x,y)$ becomes:

$$\begin{aligned}G_F(x,y) = \int\frac{d^4p}{2\pi}e^{ip(x-y)}\frac{i}{p^2-m^2+i\epsilon} &= \int\frac{d^4p}{2\pi}e^{ip(x-y)}\int_0^\infty ds e^{is(p^2-m^2+i\epsilon)} \\ &= -\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-i\left[\frac{(x-y)^2}{4s}+sm^2\right]}\end{aligned} \tag{33}$$

where in the first line, I used the identity in eq.12 to obtain an integral in Schwinger's proper-time s, and in the second line I dropped the ϵ prescription and solved the Gaussian integral in d^4p according to the following general identity for multi-dimensional Gaussian integrals: [10]

$$\int d^n p e^{-\frac{1}{2}p^\dagger\mathbf{M}p + J^\dagger p} = \sqrt{\frac{(2\pi)^n}{\det\mathbf{M}}}e^{\frac{1}{2}J^\dagger\mathbf{M}^{-1}J} \tag{34}$$

with $n=4$, $\mathbf{M} = -2isg^{\mu\nu}$ and $J = i(x-y)$.

On the other hand, the Feynman propagator from $\langle y,0|x,s\rangle$ is given by $G_F{}^{\mathrm{eff}}(x,y) = \int dse^{-ism^2}\langle y,0|x,s\rangle$ which in the limit of $\mathbf{A}\to 0$[5] becomes:

$$\lim_{\mathbf{A}\to 0}\int_0^\infty dse^{-ism^2}\langle y,0|x,s\rangle = \int_0^\infty dsC\times e^{-i[\frac{(x-y)^2}{4s}+sm^2]} \tag{35}$$

[4] It is also important to note that the last term in eq.28 $\frac{e}{2}\mathrm{tr}\{\mathbf{F}\sigma\}$ has sign inverted compared to eq.27. In fact, since $\mathbf{F}$ is anti-symmetric and σ is hermitian and unitary we know: $\mathrm{tr}\{\mathbf{F}\sigma\} = -\mathrm{tr}\{\sigma\mathbf{F}\}$.

[5] and consequently $\mathbf{F}\to 0$

where, to reduce $\langle y,0|x,s\rangle$, I used the fact that for $z\to 0$: $az\coth(z)\to a$ and $\ln(\frac{\sin(z)}{z})\to 0$.

Comparing eq.34 with eq.35, we get $C=-\frac{i}{16\pi^2 s^2}$ which makes $\langle y,0|x,s\rangle$ uniquely determined and allows us to finally have an explicit expression for $\mathcal{L}_{\rm eff}$ in terms of the electromagnetic field $\mathbf{F}$:

$$\begin{aligned}\mathcal{L}_{\rm eff}&=-\frac{1}{4}\mathbf{F}^2+\frac{i}{2}\int_0^\infty\frac{ds}{s}e^{-ism^2}\times\mathrm{Tr}\Big[\langle x|e^{-i\hat{H}s}|x\rangle\Big]+\text{const}\\ =-\frac{1}{4}\mathbf{F}^2+\frac{1}{32\pi^2}\mathrm{Tr}\Big\{&\int_0^\infty\frac{ds}{s^3}\exp\Big[-ism^2+\frac{ies}{2}\mathrm{tr}(\sigma\mathbf{F})-\frac{1}{2}\mathrm{tr}\ln\Big(\frac{\sinh(es\mathbf{F})}{es\mathbf{F}}\Big)\Big]\Big\}+\text{const}\end{aligned}\tag{36}$$

where $\langle x|e^{-i\hat{H}s}|x\rangle=\langle x,0|x,s\rangle$ which corresponds to the expression in eq.29 for $y\to x$, including the normalization constant C previously determined.

By definition of a Lagrangian density, $\mathcal{L}_{\rm eff}$ should be real except possibly near singularities.[6] To exhibit this more explicitly, Schwinger proposes to perform a deformation of the integration path which essentially consists in the substitution $s\to -is$, reducing $\mathcal{L}_{\rm eff}$ to the following form: [2]

$$\mathcal{L}_{\rm eff}=-\frac{1}{4}\mathbf{F}^2+\frac{1}{32\pi^2}\mathrm{Tr}\Big\{\int_0^\infty\frac{ds}{s^3}\exp\Big[-sm^2+\frac{es}{2}\mathrm{tr}(\sigma\mathbf{F})-\frac{1}{2}\mathrm{tr}\ln\Big(\frac{\sin(es\mathbf{F})}{es\mathbf{F}}\Big)\Big]\Big\}+\text{const}\tag{37}$$

Now, to evaluate the Dirac and normal traces within the Lagrangian density, it is useful to introduce two new quantities defined in terms of the field strengths $\vec{B}$, $\vec{E}$. [2] Specifically:

$$\mathcal{F}=\frac{1}{4}\mathbf{F}^2=\frac{1}{2}(\vec{B}^2-\vec{E}^2);\quad \mathcal{G}=\frac{1}{4}F^{\mu\nu}\tilde{F}_{\mu\nu}=\vec{E}\cdot\vec{B}\tag{38}$$

where $\tilde{F}_{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. Using the identity on the anti-commutator between the pauli matrices $\frac{1}{2}\{\sigma_{\mu\nu},\sigma_{\lambda k}\}=\delta_{\mu\lambda}\delta_{\nu k}-\delta_{\mu k}\delta_{\nu\lambda}+i\epsilon_{\mu\nu\lambda k}\Upsilon_5$ one can derive that: [2]

$$[\mathrm{tr}(\sigma\mathbf{F})]^2=(\sigma_{\mu\nu}F^{\mu\nu})^2=2(F^{\mu\nu})^2\mathbf{1}+i\Upsilon_5\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}F_{\alpha\beta}=8(\mathcal{F}+\Upsilon_5\mathcal{G})\tag{39}$$

Then, since $\Upsilon_5^2=-1$,[7] $\mathrm{Tr}(\sigma\mathbf{F})$ has four eigenvalues $\lambda_i=\pm\sqrt{8(\mathcal{F}\pm i\mathcal{G})}$ given by all the 4 possible sign combinations. Thus:

$$\begin{aligned}\mathrm{Tr}\Big[e^{\frac{es}{2}\mathrm{tr}(\sigma\mathbf{F})}\Big]=2\cosh\Big[es\sqrt{2\mathcal{F}+i\mathcal{G}}\Big]+2\cosh\Big[es\sqrt{2\mathcal{F}-i\mathcal{G}}\Big]&=4\Re\cosh[esX]\\ \text{where } X=\sqrt{2\mathcal{F}+i\mathcal{G}}&=\sqrt{(\vec{B}+i\vec{E})^2}\end{aligned}\tag{40}$$

Next, we need to express $\frac{1}{2}\mathrm{tr}\Big[\ln\Big(\frac{\sin(es\mathbf{F})}{es\mathbf{F}}\Big)\Big]$ in terms of its eigenvalues which are in turn determined from those of a constant $F_{\mu\nu}$. [10] To find the eigenvalues of $F_{\mu\nu}$, it is useful to introduce two trivial relations which come strictly from the definition of $\mathcal{G}$ and $\mathcal{F}$: [2]

$$F^{\mu\lambda}\tilde{F}_{\lambda\nu}=-\delta^\mu_\nu\mathcal{G};\quad \tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu}-F^{\mu\lambda}F_{\lambda\nu}=2\delta^\mu_\nu\mathcal{F}\tag{41}$$

From the eigenvalue equations $F_{\mu\nu}v_\nu=\lambda^F v_\mu$ and $\tilde{F}_{\mu\nu}v_\nu=-\frac{\mathcal{G}}{\lambda^F}v_\mu$,[8] we get by iteration two more relations:

$$F^{\mu\lambda}F_{\lambda\nu}v_\nu=(\lambda^F)^2v_\mu;\quad \tilde{F}^{\mu\lambda}\tilde{F}_{\lambda\nu}v_\nu=\frac{\mathcal{G}^2}{(\lambda^F)^2}v_\mu\tag{42}$$

[6] This is a very important feature of $\mathcal{L}_{\rm eff}$. Indeed, as I will show in more detail in sec.5, it is the fact that $\mathcal{L}_{\rm eff}$ picks up an imaginary part near its singularities that explains the non-zero probability for pair-production to occur.

[7] Υ_5 has eigenvalues ± 1

[8] this equation arises from the 1st equation in 41

which, when plugged in the 2nd equation from 41, yields the eigenvalue equation:

$$
\begin{aligned}
&(\lambda^F)^4 + 2\mathcal{F}(\lambda^F)^2 - \mathcal{G}^2 = 0 \\
&\Rightarrow \lambda^F{}^+_\pm = \pm\frac{i}{\sqrt{2}}\Big[\sqrt{\mathcal{F}+i\mathcal{G}} + \sqrt{\mathcal{F}-i\mathcal{G}}\Big] \\
&\Rightarrow \lambda^F{}^-_\pm = \pm\frac{i}{\sqrt{2}}\Big[\sqrt{\mathcal{F}+i\mathcal{G}} - \sqrt{\mathcal{F}-i\mathcal{G}}\Big]
\end{aligned}
\tag{43}
$$

$\frac{1}{2}\mathrm{tr}\Big[\ln\Big(\frac{\sin(es\mathbf{F})}{es\mathbf{F}}\Big)\Big]$ expressed in terms of the 4 eigenvalues λ_i^F equals $\ln\Big(\sqrt{\prod_{i=1}^4 g(es\lambda_i^F)}\Big)$ where the function $g(x) = \frac{\sinh(x)}{x}$. Hence after some trivial algebra steps one gets:

$$
\exp\Big\{-\frac{1}{2}\mathrm{tr}\Big[\ln\Big(\frac{\sin(es\mathbf{F})}{es\mathbf{F}}\Big)\Big]\Big\} = \frac{e^2 s^2 \mathcal{G}}{\Im\cosh(esX)}
\tag{44}
$$

where X is the same as defined in eq.38.

Putting everything together we obtain the final result for $\mathcal{L}_{\text{eff}}$ which takes the form of the so called Euler-Heisenberg Lagrangian $\mathcal{L}_{EH}$:

$$
\mathcal{L}_{EH} = -\frac{1}{4}\mathbf{F}^2 - \frac{e^2}{8\pi^2}\int_0^\infty \frac{ds}{s} e^{-sm^2}\mathcal{G}\frac{\Re\cosh(esX)}{\Im\cosh(esX)} + \text{const}
\tag{45}
$$

5 Pair Production Rate

To investigate and then compute the rate of pair-production one must look into the behaviour of $\mathcal{L}_{EH}$ near its singularities. However, $\mathcal{L}_{EH}$ in eq.4 is the unrenormalized Euler-Heisenberg effective Lagrangian, thus, as a preliminary step, we need to normalize the Lagrangian. The simplest way here is to use minimal subtraction, expanding the integrand in e up to the two leading terms: [10]

$$
\frac{\Re\cosh(esX)}{\Im\cosh(esX)} = -\frac{4}{e^2 s^2} - \frac{2}{3}\mathbf{F}^2 + \dots
\tag{46}
$$

These terms result in a UV divergence for $s \to 0$. In this case, these divergences can be regulated by simply applying a cut off at $s > s_0$. [10] The required counter-terms are then the two leading terms from eq.46 with opposite sign. Moreover, to normalize the infrared divergence for $s \to \infty$ we must also reinsert the Feynman prescription such that $sm^2 \to sm^2 - i\epsilon s$ with $\lll 1$. [11] Putting all together, we can rewrite the normalized $\mathcal{L}_{EH}$ in the form:

$$
\mathcal{L}_{EH} = -\frac{1}{4}\mathbf{F}^2 - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s} e^{-sm^2}\Big[\mathcal{G}\frac{\Re\cosh(esX)}{\Im\cosh(esX)} - \frac{4}{e^2 s^2} - \frac{1}{3}\mathbf{F}^2\Big]
\tag{47}
$$

Now we are finally ready to explore the case of a constant electric field: $\vec{E} = \text{cost}$; $\vec{B} = 0$. Generally we know from their definition that: $\mathbf{F}^2 = 2(\vec{B}^2 - \vec{E}^2)$ and $X^2 = (\vec{B} + i\vec{E})^2$. Hence for the case of a constant electric field ($|\vec{E}| = E$): $\mathbf{F}^2 \to -2E^2$ and $X \to i\vec{E}$. According to these relations, the Euler-Heisenberg Lagrangian simplifies to:

$$
\mathcal{L}_{EH} = \frac{1}{2}E^2 - \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3} e^{-sm^2}\Big[eEs\cot(esE) - 1 + \frac{1}{3}(eEs)^2\Big]
\tag{48}
$$

In this form, we can see that the integrand in $\mathcal{L}_{EH}$ has poles for real E at $s_n = \frac{n\pi}{eE}$ for $n = 1, 2 \dots$. Thus the integral in ds can be computed using the residues method. It must be at first transformed into a integral over

$(-\infty, +\infty)$, then its integration contour must be deformed to include the negative imaginary axis and pick up the contribution of the poles from the coth function. [11] The evaluation of this integral results in a positive imaginary contribution to $\mathcal{L}_{EH}$ which is precisely:

$$\begin{aligned}\Im\mathcal{L}_{EH} &= \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}e^{-sm^2}[eEs\cot(esE)] = \frac{1}{16\pi^2}\int_{-\infty}^\infty \frac{ds}{s^3}e^{-sm^2}[eEs\cot(esE)] \\ &= \frac{1}{8\pi^2}\times\pi\sum_{n=1}^\infty \mathrm{Res}[f(s), s_n] = \frac{1}{8\pi^2}\times\pi\sum_{n=1}^\infty \frac{1}{s_n^2}e^{-m^2 s_n} \\ &= \frac{e^2E^2}{2\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}e^{-\frac{n\pi m^2}{eE}}\end{aligned} \tag{49}$$

where $f(s) = \frac{e^{-sm^2}}{s^2}[eE\cot(esE)]$.

As I explained in sec.3, introducing the effective action Υ was done in order to express the vacuum persistence probability $P_0 = S_0^2$ using the fact that $S_0 = e^{i\Upsilon}$. $|e^{i\Upsilon}|^2$ therefore measures the probability that no pairs are produced over time T and in a volume V. We then have:

$$P_0 = |e^{i\Upsilon}|^2 = e^{i\Upsilon}e^{-i\Upsilon^*} = e^{-2\Im[\Upsilon]} = e^{-2VT\Im\mathcal{L}_{EH}} \tag{50}$$

where $\Im\mathcal{L}_{EH}$ is as computed in eq.49. Finally, according to my discussion at the beginning of sec.3, the electron-positron pair production per volume per unit time Γ is exactly equal to $2\Im\mathcal{L}_{EH}$:

$$\Gamma = 2\Im\mathcal{L}_{EH} = \frac{e^2E^2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}e^{-\frac{n\pi m^2}{eE}} \tag{51}$$

in agreement with eq.1.

6 Conclusion

To conclude, in this research project, we saw how, in the presence of a constant electric field, the probability of spontaneous electron-positron pair production in vacuum is non-zero. Quantum mechanically, this can be explained as a tunneling phenomenon by which a particle from the Dirac sea is pulled into the positive energy states. In terms of the in-out formalism of quantum field theory, the Schwinger effect arises from the existence of poles in the proper-time representation of the effective action. In fact, their existence implies that the one-loop effective action has not only a real part, the vacuum polarization, but also an imaginary part, representing the vacuum persistence. Finally, this effect also turns out to be non-perturbative, as demonstrated by the $\frac{1}{eE}$ dependence in the exponential of Γ, which means that this result cannot be obtained from the standard perturbation theory and the Feynman diagram formalism for quantum electrodynamics.

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